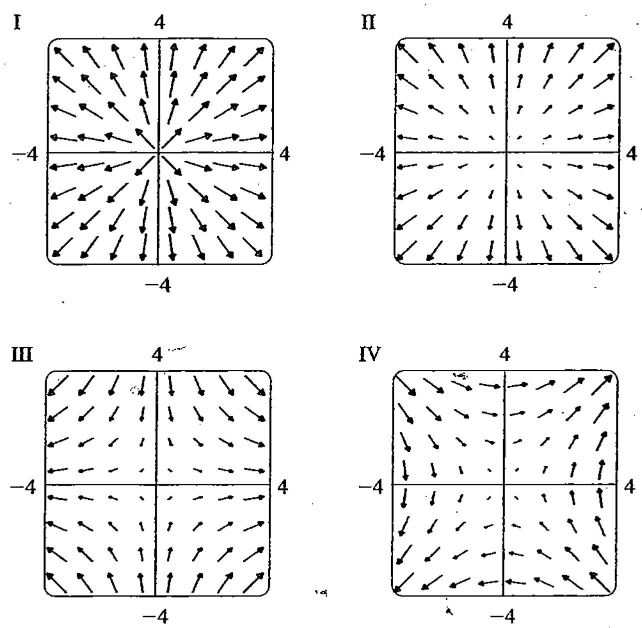
1. (8 points) Match the vector fields $\mathbf{F}$ with the pictures I–IV:

(a) $\mathbf{F}(x, y) = \langle y, x \rangle$ IV

(b) $\mathbf{F}(x, y) = \langle 2x, -2y \rangle$ III

(c) $\mathbf{F}(x, y) = \langle 2x, 2y \rangle$ II

(d) $\mathbf{F}(x, y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$ I



2. (8 points) Are the following vector fields $\mathbf{F}$ conservative? If so find a function f with $\nabla f = \mathbf{F}$, and otherwise give a reason why not.

(a) $\mathbf{F}(x, y) = \langle 3x \sin y, 2x^2 \cos y \rangle$

$P \quad Q$

$$\frac{\partial P}{\partial y} = 3x \cos y \qquad \frac{\partial Q}{\partial x} = 4x \cos y$$

not equal, so $\underline{F}$ is not conservative

(b) $\mathbf{F}(x, y) = \langle y^2 \cos x - 3, 2y \sin x \rangle$

$P \quad Q$

$$\frac{\partial P}{\partial y} = 2y \cos x \qquad \frac{\partial Q}{\partial x} = 2y \cos x$$ ✓

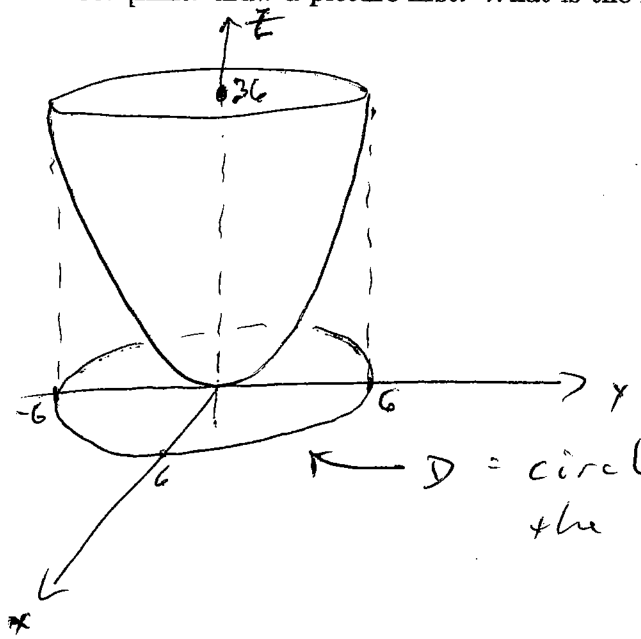
Also $\underline{F}$ is defined on $\mathbb{R}^2$. $\underline{F}$ is conservative.

$$f_x = y^2 \cos x - 3 \Rightarrow f = y^2 \sin x - 3x + C(y)$$

$$f_y = 2y \sin x \Rightarrow f = y^2 \sin x + D(x)$$

So $f(x,y) = y^2 \sin x - 3x + E$

3. (8 points) Find the surface area of the part of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 36$. [Hint: draw a picture first. What is the region in the xy–plane?]



$$z = f(x,y) = x^2 + y^2$$
$$f_x = 2x$$
$$f_y = 2y$$

D = circle of radius 6 in the x-y plane

$$SA = \iint_D \sqrt{1 + (2x)^2 + (2y)^2}\, dA$$

↓ polar coords

$$= \int_0^{2\pi} \int_0^{6} \sqrt{1 + 4r^2\cos^2\theta + 4r^2\sin^2\theta}\; r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{6} \sqrt{1 + 4r^2}\; r\, dr\, d\theta$$

$$u = 1 + 4r^2$$
$$du = 8r\, dr$$

$$= \int_0^{2\pi} \int_1^{145} \frac{1}{8} u^{1/2}\, du\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{12} u^{3/2} \Big|_1^{145}\, d\theta$$

$$= 2\pi \left(\frac{1}{12}\left(145^{3/2} - 1\right)\right) = \boxed{\frac{\pi}{6}\left(145^{3/2} - 1\right)}$$

4. (8 points) Evaluate the following line integrals:

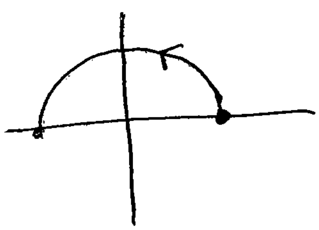
(a) $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x,y) = \langle y^2, x^2y \rangle$ and C is given by $\mathbf{r}(t) = \langle t^2, t^3 \rangle$, $0 \leq t \leq 3$.

$$\int_0^3 \langle (t^3)^2, (t^2)^2 t^3 \rangle \cdot \langle 2t, 3t^2 \rangle \, dt$$

$$= \int_0^3 2t^7 + 3t^9 \, dt = \frac{1}{4}t^8 + \frac{3}{10}t^{10} \Big|_0^3$$

$$= \boxed{\frac{3^8}{4} + \frac{3^{11}}{10}}$$

(b) $\int_C xy^2 \, ds$ where C is the upper half of the unit circle $x^2 + y^2 = 1$.



$$r(t) = \langle \cos t, \sin t \rangle, \quad 0 \leq t \leq \pi$$

$$r'(t) = \langle -\sin t, \cos t \rangle$$

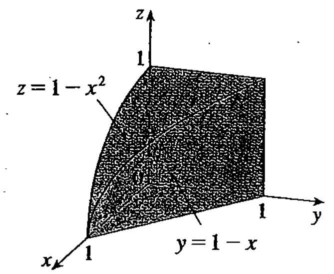
$$ds = \sqrt{(-\sin t)^2 + (\cos t)^2} \, dt = dt$$

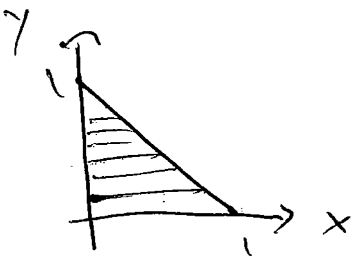
$$\int_C xy^2 ds = \int_0^\pi \cos t \sin^2 t \, dt$$

$$u = \sin t, \; du = \cos t \, dt$$

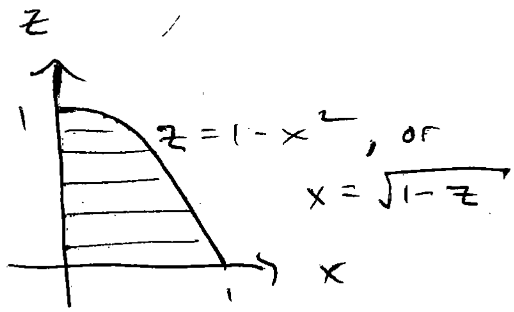
$$\int_0^0 u^2 du = \boxed{0}$$

5. (8 points) The region E is bounded by the surface $z = 1 - x^2$, the plane $y = 1 - x$, and the three coordinate planes. Express the integral $\iiint_E f(x,y,z)\,dV$ in three different ways, using $dV = dz\,dx\,dy$, $dV = dy\,dx\,dz$, and $dV = dy\,dz\,dx$.

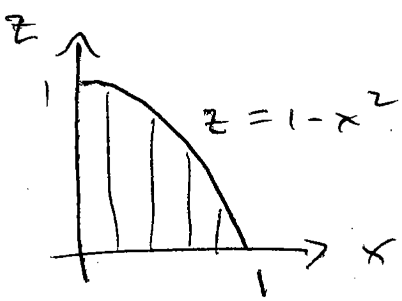




$$\boxed{\int_0^1 \int_0^{1-y} \int_0^{1-x^2} f(x,y,z)\,dz\,dx\,dy}$$

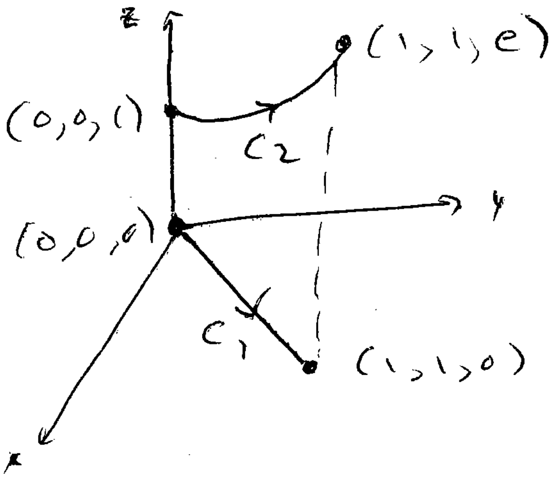


$$\boxed{\int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} f(x,y,z)\,dy\,dx\,dz}$$



$$\boxed{\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z)\,dy\,dz\,dx}$$

6. (6 points) Let $f(x, y, z) = \cos(x^2 + 2y - z)$. Let C_1 be the line segment from $(0, 0, 0)$ to $(1, 1, 0)$ and let C_2 be the curve on the surface $z = e^{xy}$ lying directly above C_1. Find $\int_{C_1} \nabla f \cdot d\mathbf{r}$ and $\int_{C_2} \nabla f \cdot d\mathbf{r}$.



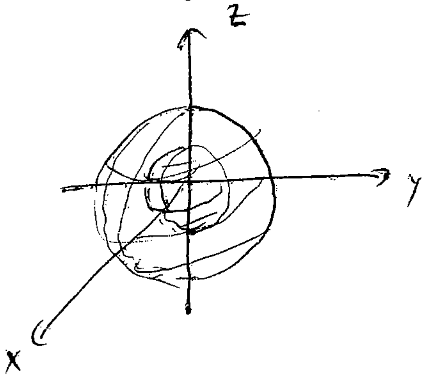
$$\int_{C_1} \nabla f \cdot d\underline{r} = f(1,1,0) - f(0,0,0)$$

$$= \boxed{\cos(3) - 1}.$$

$$\int_{C_2} \nabla f \cdot d\underline{r} = f(1,1,e) - f(0,0,1)$$

$$= \boxed{\cos(3-e) - \cos(-1)}$$

7. (8 points) Evaluate $\iiint_E z\, dV$ where E is the region between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$ and in front of the yz–plane (that is, in the region where $x \geq 0$).



spherical coords:

$$E = \begin{cases} 1 \le \rho \le 2 \\ -\pi/2 \le \theta \le \pi/2 \\ 0 \le \varphi \le \pi \end{cases}$$

$$\int_0^{\pi} \int_{-\pi/2}^{\pi/2} \int_1^2 (\rho \cos\varphi)\rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi$$

$u = \sin\varphi$
$du = \cos\varphi \, d\varphi$

$$= \int_0^{\pi} \cos\varphi \sin\varphi \, d\varphi \int_{-\pi/2}^{\pi/2} d\theta \int_1^2 \rho^3 \, d\rho$$

$$= \int_0^0 u\, du \; (\pi) \left(\tfrac{1}{4}\rho^4 \Big|_1^2 \right)$$

$$= (0)\,(\pi)\left(\tfrac{1}{4}(2^4 - 1)\right) = \boxed{0}.$$