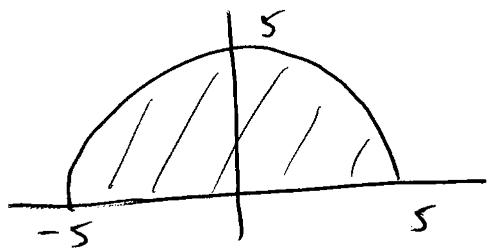
1. (6 points) Evaluate $\iint_R \cos(x^2+y^2)\,dA$ by changing to polar coordinates. Here R is the region that lies above the x-axis and within the circle $x^2+y^2=25$.



$R: 0 \le r \le 5, \quad 0 \le \theta \le \pi$

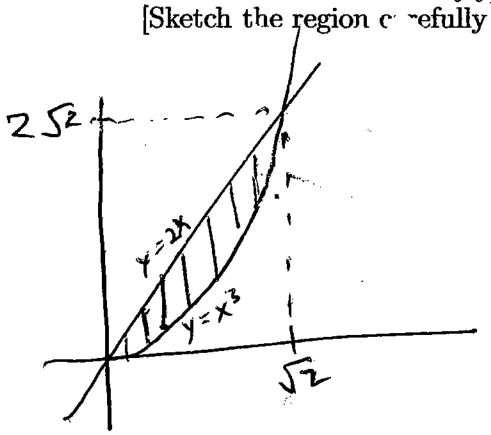
$$\int_0^{\pi}\int_0^{5} \cos(r^2)\, r\,dr\,d\theta \qquad \left[\begin{array}{l} u=r^2 \\ du = 2r\,dr \end{array}\right]$$

$$= \int_0^{\pi}\int_0^{25} \tfrac{1}{2}\cos(u)\,du = \int_0^{\pi}\left(\tfrac{1}{2}\sin(u)\Big|_0^{25}\right) d\theta$$

$$= \int_0^{\pi} \tfrac{1}{2}\sin(25)\,d\theta$$

$$= \boxed{\tfrac{\pi}{2}\sin(25)}$$

2. (6 points) Evaluate $\iint_D (x+y)\,dA$ where D is the region bounded by $y=2x$ and $y=x^3$. [Sketch the region carefully first.]



$2x = x^3 \Rightarrow x^3 - 2x = 0$

$x(x^2-2) = 0$

$x = 0, \pm\sqrt{2}$

$$\int_0^{\sqrt{2}}\int_{x^3}^{2x} (x+y)\,dy\,dx = \int_0^{\sqrt{2}} \left(xy + \tfrac{1}{2}y^2 \Big|_{x^3}^{2x} \right) dx$$

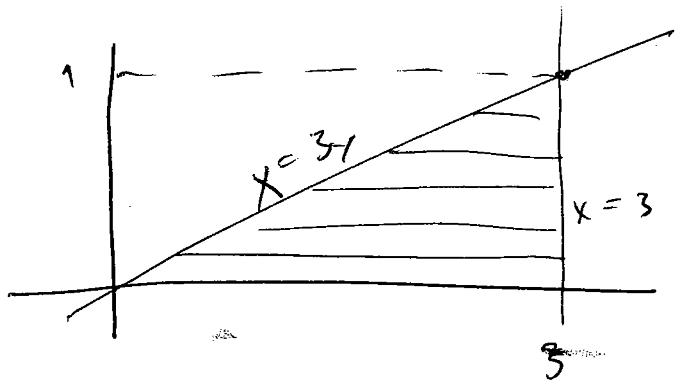
$$= \int_0^{\sqrt{2}} 2x^2 - x^4 + \tfrac{1}{2}(2x)^2 - \tfrac{1}{2}(x^3)^2\,dx$$

$$= \int_0^{\sqrt{2}} 4x^2 - x^4 - \tfrac{1}{2}x^6\,dx = \tfrac{4}{3}x^3 - \tfrac{1}{5}x^5 - \tfrac{1}{14}x^7 \Big|_0^{\sqrt{2}}$$

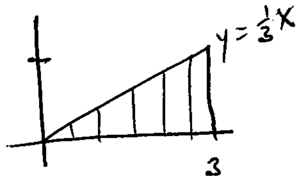
$$= \boxed{\tfrac{4}{3}2^{3/2} - \tfrac{1}{5}2^{5/2} - \tfrac{1}{14}2^{7/2}}$$

[note – if you include left region as well, the integral is 0.]

3a. (3 points) Sketch carefully the region for the integral $\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$.



3b. (5 points) Change the order of integration and evaluate the integral.

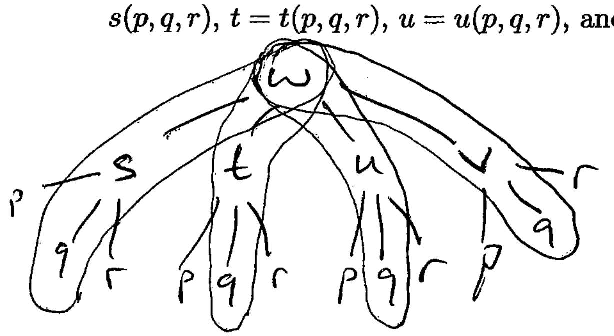


$$\int_0^3 \int_0^{\frac{1}{3}x} e^{x^2} dy\, dx = \int_0^3 \left(e^{x^2} y \Big|_0^{\frac{1}{3}x} \right) dx$$

$$= \int_0^3 \frac{1}{3} x e^{x^2} dx \quad \begin{bmatrix} u = x^2 \\ du = 2x\,dx \end{bmatrix} = \int_0^9 \frac{1}{6} e^u \, du$$

$$= \frac{1}{6} e^u \Big|_0^9 = \boxed{\frac{1}{6}\left(e^9 - 1\right)}$$

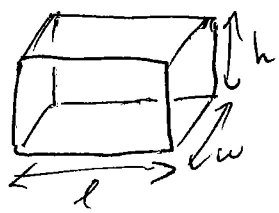
4. (4 points) Write down a tree diagram and the chain rule for $\frac{\partial w}{\partial q}$ when $w = f(s,t,u,v)$, $s = s(p,q,r)$, $t = t(p,q,r)$, $u = u(p,q,r)$, and $v = v(p,q,r)$.



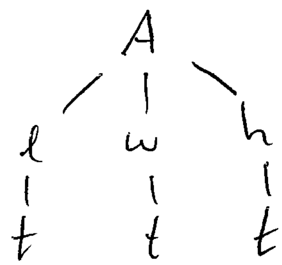
$$\boxed{\frac{\partial w}{\partial q} = \frac{\partial w}{\partial s}\frac{\partial s}{\partial q} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial q} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial q} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial q}}$$

5. (8 points) The length ℓ, width w, and height h of a box change with time. At a certain instant the dimensions are $\ell = 2$ m, $w = 1$ m, $h = 3$ m. Also ℓ and w are increasing at a rate of 4 m/s and h is decreasing at a rate of 5 m/s.

(a) Write a formula for the surface area A, and then write down the chain rule for $\frac{dA}{dt}$.



$$\boxed{A = 2\ell w + 2\ell h + 2wh}$$



$$\boxed{\frac{dA}{dt} = \frac{\partial A}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial A}{\partial w}\frac{dw}{dt} + \frac{\partial A}{\partial h}\frac{dh}{dt}}$$

(b) At this instant find the rate at which the surface area is changing.

$$\frac{\partial A}{\partial \ell} = 2w + 2h, \quad \frac{\partial A}{\partial w} = 2\ell + 2h, \quad \frac{\partial A}{\partial h} = 2\ell + 2w$$

At this instant:

$$\frac{\partial A}{\partial \ell} = 8 \text{ m}^2/\text{s}, \quad \frac{\partial A}{\partial w} = 10 \text{ m}^2/\text{s}, \quad \frac{\partial A}{\partial h} = 6 \text{ m}^2/\text{s}$$

So $$\frac{dA}{dt} = (8)(4) + (10)(4) + (6)(-5)$$

$$= \boxed{42 \text{ m}^2/\text{s}}$$

6a. (4 points) A moth at $(3, 1, 2)$ wants to get warm as quickly as possible. The temperature is given by $T(x, y, z) = \frac{x}{z} + 2x^2y$. In which direction should the moth fly?

$\nabla T = \langle \frac{1}{z} + 4xy, 2x^2, \frac{-x}{z^2} \rangle$

at $(3,1,2)$ it is $\langle 12\frac{1}{2}, 18, \frac{-3}{4} \rangle$.

The moth should fly in the direction $\langle 12\frac{1}{2}, 18, -\frac{3}{4} \rangle$.

6b. (5 points) Use the gradient to find the directional derivative of $f(x, y) = x^2 \sin y - y^2$ at the point $(1, \frac{\pi}{2})$ in the direction of the origin. You do not need to simplify your answer.

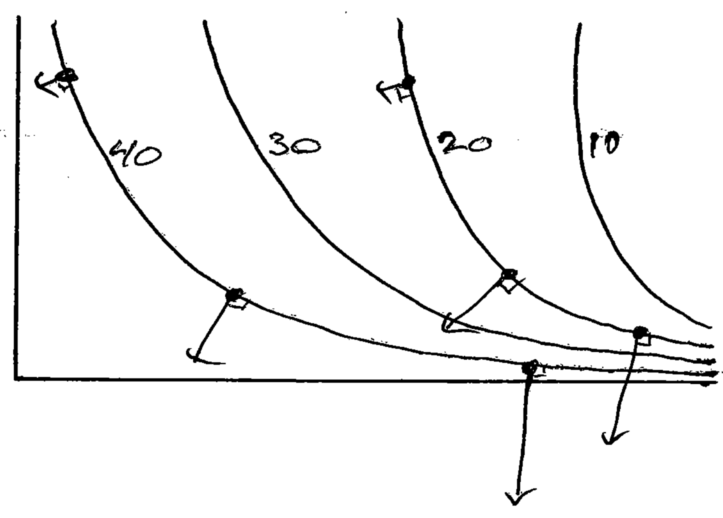
$\nabla f = \langle 2x \sin y, x^2 \cos y - 2y \rangle$

at $(1, \pi/2)$ it is $\langle 2, -\pi \rangle$.

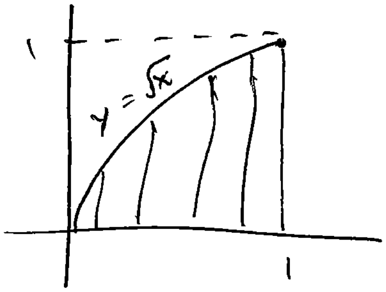
Direction is $\underline{u} = \dfrac{\langle -1, -\pi/2 \rangle}{|\langle -1, -\pi/2 \rangle|} = \dfrac{1}{\sqrt{1+\pi^2/4}} \langle -1, -\pi/2 \rangle$.

$D_{\underline{u}} f(1, \frac{\pi}{2}) = \langle 2, -\pi \rangle \cdot \underline{u} = \boxed{\dfrac{1}{\sqrt{1+\pi^2/4}} \left(-2 + \dfrac{\pi^2}{2}\right)}$.

6c. (5 points) The picture below shows level curves of a function $f(x, y)$. Draw the gradient vectors at the indicated points. [Keep in mind the relative lengths of the vectors.]



7. (8 points) A lamina occupies the region D bounded by $y = 0$, $x = 1$, and $y = \sqrt{x}$, and has density $\rho(x, y) = x$. Find the mass and the center of mass of the lamina.



$$\text{mass} = \iint_D \rho(x,y)\,dA$$

$$= \int_0^1 \int_0^{\sqrt{x}} x\,dy\,dx = \int_0^1 \left(xy \Big|_0^{\sqrt{x}} \right) dx$$

$$= \int_0^1 x^{3/2}\,dx = \frac{2}{5} x^{5/2} \Big|_0^1 = \boxed{\frac{2}{5}}.$$

$$\bar{x} = \frac{5}{2} \int_0^1 \int_0^{\sqrt{x}} x^2\,dy\,dx = \frac{5}{2} \int_0^1 \left(x^2 y \Big|_0^{\sqrt{x}} \right) dx = \frac{5}{2} \int_0^1 x^{5/2}\,dx$$

$$= \frac{5}{2} \left(\frac{2}{7} x^{7/2} \Big|_0^1 \right) = \frac{5}{7}.$$

$$\bar{y} = \frac{5}{2} \int_0^1 \int_0^{\sqrt{x}} xy\,dy\,dx = \frac{5}{2} \int_0^1 \left(\frac{1}{2} xy^2 \Big|_0^{\sqrt{x}} \right) dx$$

$$= \frac{5}{4} \int_0^1 x^2\,dx = \frac{5}{4} \left(\frac{1}{3} x^3 \Big|_0^1 \right) = \frac{5}{12}.$$

$$\text{Center of mass} = (\bar{x}, \bar{y}) = \boxed{\left(5/7, 5/12 \right)}.$$