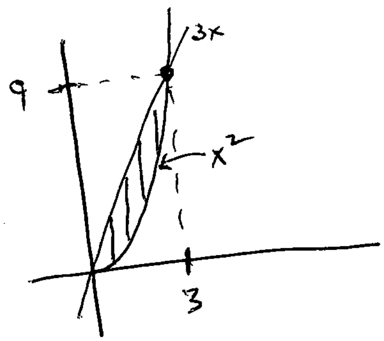
1. (6 points) Evaluate $\iint_D (x+y)\,dA$ where D is the region bounded by $y = 3x$ and $y = x^2$. [Sketch the region carefully first.]



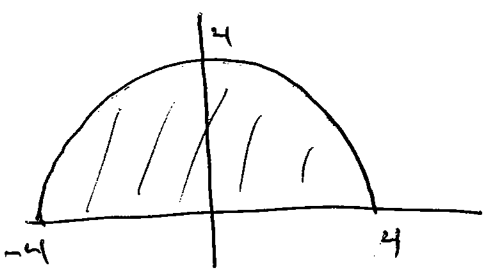
$$\int_0^3 \int_{x^2}^{3x} (x+y)\,dy\,dx = \int_0^3 \left(xy + \tfrac{1}{2}y^2 \Big|_{x^2}^{3x} \right) dx$$

$$= \int_0^3 3x^2 + \tfrac{9}{2}x^2 - x^3 - \tfrac{1}{2}x^4\,dx$$

$$= x^3 + \tfrac{3}{2}x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{10}x^5 \Big|_0^3$$

$$= 27 + \tfrac{3}{2}(27) - \tfrac{1}{4}(81) - \tfrac{1}{10}(243) = \boxed{22.95}$$

2. (6 points) Evaluate $\iint_R \cos(x^2+y^2)\,dA$ by changing to polar coordinates. Here R is the region that lies above the x-axis and within the circle $x^2+y^2 = 16$.



$R: \quad 0 \le r \le 4, \quad 0 \le \theta \le \pi$

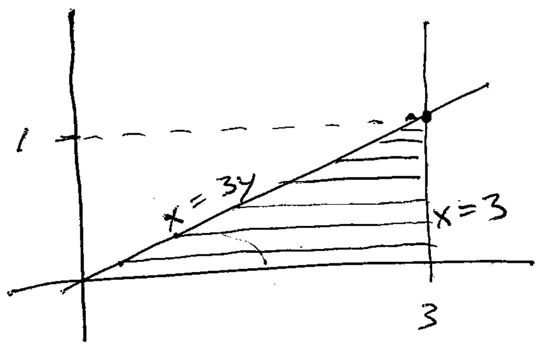
$$\int_0^\pi \int_0^4 \cos(r^2)\, r\,dr\,d\theta$$

$u = r^2$
$du = 2r\,dr$

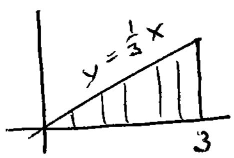
$$\int_0^\pi \int_0^{16} \tfrac{1}{2}\cos(u)\,du\,d\theta = \int_0^\pi \left(\tfrac{1}{2}\sin(u) \Big|_0^{16} \right) d\theta = \int_0^\pi \tfrac{1}{2}\sin(16)\,d\theta$$

$$= \boxed{\frac{\pi}{2}\sin(16)}$$

3a. (3 points) Sketch carefully the region for the integral $\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$.



3b. (5 points) Change the order of integration and evaluate the integral.

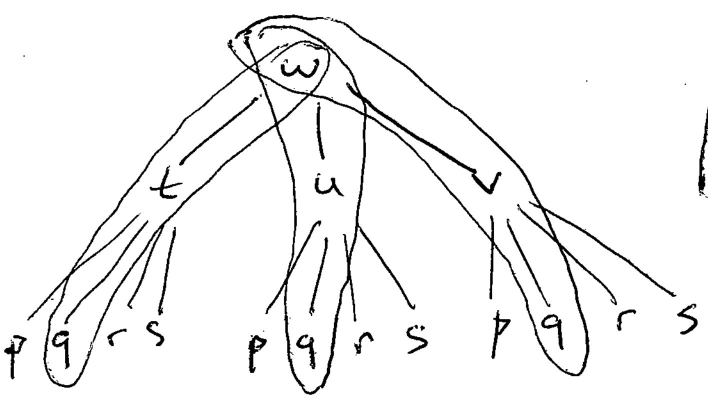


$$\int_0^3 \int_0^{\frac{1}{3}x} e^{x^2}\, dy\, dx = \int_0^3 \left(e^{x^2} y \Big|_0^{\frac{1}{3}x} \right) dx$$

$$= \int_0^3 \frac{1}{3} x e^{x^2}\, dx \quad \begin{bmatrix} u = x^2 \\ du = 2x\,dx \end{bmatrix} \quad = \int_0^9 \frac{1}{6} e^u\, du$$

$$= \frac{1}{6} e^u \Big|_0^9 = \boxed{\frac{1}{6}(e^9 - 1)}.$$

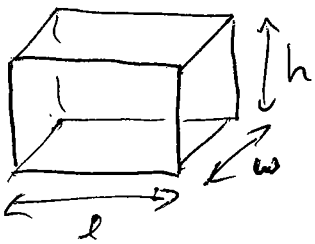
4. (4 points) Write down a tree diagram and the chain rule for $\frac{\partial w}{\partial q}$ when $w = f(t,u,v)$, $t = t(p,q,r,s)$, $u = u(p,q,r,s)$, and $v = v(p,q,r,s)$.



$$\boxed{\frac{\partial w}{\partial q} = \frac{\partial w}{\partial t}\frac{\partial t}{\partial q} + \frac{\partial w}{\partial u}\frac{\partial u}{\partial q} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial q}}$$

5. (8 points) The length ℓ, width w, and height h of a box change with time. At a certain instant the dimensions are $\ell = 2$ m, $w = 1$ m, $h = 3$ m. Also ℓ and w are increasing at a rate of 3 m/s and h is decreasing at a rate of 4 m/s.

(a) Write a formula for the surface area A, and then write down the chain rule for $\frac{dA}{dt}$.



$$\boxed{A = 2\ell w + 2wh + 2\ell h}$$

$$\begin{array}{ccc} & A & \\ \ell & w & h \\ | & | & | \\ t & t & t \end{array}$$

$$\boxed{\frac{dA}{dt} = \frac{\partial A}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial A}{\partial w}\frac{dw}{dt} + \frac{\partial A}{\partial h}\frac{dh}{dt}}$$

(b) At this instant find the rate at which the surface area is changing.

$$\frac{\partial A}{\partial \ell} = 2w + 2h \,, \quad \frac{\partial A}{\partial w} = 2\ell + 2h \,, \quad \frac{\partial A}{\partial h} = 2\ell + 2w$$

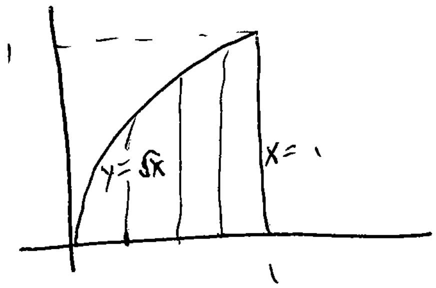
At this instant:

$$\frac{\partial A}{\partial \ell} = 8 \text{ m}^2/\text{s} \,, \quad \frac{\partial A}{\partial w} = 10 \text{ m}^2/\text{s} \,, \quad \frac{\partial A}{\partial h} = 6 \text{ m}^2/\text{s}$$

So $$\frac{dA}{dt} = (8)(3) + (10)(3) + (6)(-4) \quad \text{m}^2/\text{s}$$

$$= \boxed{30 \text{ m}^2/\text{s}} .$$

6. (8 points) A lamina occupies the region D bounded by $y = 0$, $x = 1$, and $y = \sqrt{x}$, and has density $\rho(x, y) = x$. Find the mass and the center of mass of the lamina.



$$\text{mass} = \iint_D \rho(x,y)\, dA$$

$$= \int_0^1 \int_0^{\sqrt{x}} x\, dy\, dx = \int_0^1 \left(xy \Big|_0^{\sqrt{x}} \right) dx$$

$$= \int_0^1 x^{3/2}\, dx = \frac{2}{5} x^{5/2} \Big|_0^1 = \boxed{\frac{2}{5}}.$$

$$\bar{x} = \frac{5}{2} \int_0^1 \int_0^{\sqrt{x}} x^2\, dy\, dx = \frac{5}{2} \int_0^1 \left(x^2 y \Big|_0^{\sqrt{x}} \right) dx = \frac{5}{2} \int_0^1 x^{5/2}\, dx$$

$$= \frac{5}{2} \left(\frac{2}{7} x^{7/2} \Big|_0^1 \right) = \frac{5}{7}.$$

$$\bar{y} = \frac{5}{2} \int_0^1 \int_0^{\sqrt{x}} xy\, dy\, dx = \frac{5}{2} \int_0^1 \left(\frac{1}{2} x y^2 \Big|_0^{\sqrt{x}} \right) dx = \frac{5}{4} \int_0^1 x^2\, dx$$

$$= \frac{5}{4} \left(\frac{1}{3} x^3 \Big|_0^1 \right) = \frac{5}{12}.$$

Center of mass $= (\bar{x}, \bar{y}) = \boxed{\left(5/7,\ 5/12 \right)}.$

7a. (4 points) A moth at $(1, 2, 2)$ wants to get warm as quickly as possible. The temperature is given by $T(x, y, z) = \frac{x}{z} + 2x^2y$. In which direction should the moth fly?

$\nabla T = \langle \frac{1}{z} + 4xy, 2x^2, \frac{-x}{z^2} \rangle$

at $(1,2,2)$ it is $\langle \frac{1}{2} + 8, 2, \frac{-1}{4} \rangle$.

The moth should fly in the direction $\langle 8\frac{1}{2}, 2, -\frac{1}{4} \rangle$.

7b. (5 points) Use the gradient to find the directional derivative of $f(x, y) = x^2 \sin y - y^2$ at the point $(1, \frac{\pi}{2})$ in the direction of the origin. You do not need to simplify your answer.

$\nabla f = \langle 2x \sin y, x^2 \cos y - 2y \rangle$

at $(1, \frac{\pi}{2})$ it is $\langle 2, -\pi \rangle$. Direction is

$\underline{u} = \dfrac{\langle -1, -\frac{\pi}{2} \rangle}{|\langle -1, -\frac{\pi}{2} \rangle|} = \left\langle \dfrac{-1}{\sqrt{1 + \frac{\pi^2}{4}}}, \dfrac{-\pi/2}{\sqrt{1 + \frac{\pi^2}{4}}} \right\rangle$.

$D_u f = \langle 2, -\pi \rangle \cdot \underline{u} = \boxed{\dfrac{-2}{\sqrt{1 + \pi^2/4}} + \dfrac{\pi^2/2}{\sqrt{1 + \pi^2/4}}}$.

7c. (5 points) The picture below shows level curves of a function $f(x, y)$. Draw the gradient vectors at the indicated points. [Keep in mind the relative lengths of the vectors.]