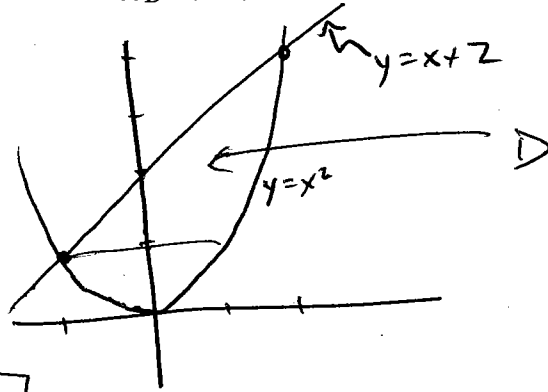


1. (8 points) Let D be the region in the xy -plane between the curves $y = x^2$ and $y = x + 2$. Draw the region D carefully. Then set up the integral $\iint_D f(x, y) dA$ in two different ways, using $dA = dx dy$ and $dA = dy dx$.

intersections:

$$\begin{aligned} x^2 &= x + 2 \\ x^2 - x - 2 &= 0 \\ (x - 2)(x + 1) &= 0 \\ x &= -1, 2 \end{aligned}$$



$$\int_{-1}^2 \int_{x^2}^{x+2} f(x, y) dy dx$$

$$\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y) dx dy + \int_1^4 \int_{y-2}^{\sqrt{y}} f(x, y) dx dy$$

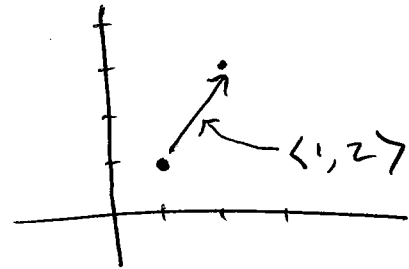
2. (8 points) Use the gradient to find the directional derivative of $f(x, y) = x^4 y^3 - 3x^2$ at the point $(1, 1)$ in the direction of the point $(2, 3)$.

$$\nabla f = \langle 4x^3 y^3 - 6x, 3x^4 y^2 \rangle$$

$$\nabla f(1, 1) = \langle -2, 3 \rangle$$

$$\underline{u} = \frac{\langle 1, 2 \rangle}{|\langle 1, 2 \rangle|} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

$$D_{\underline{u}} f(1, 1) = \nabla f(1, 1) \cdot \underline{u} = \frac{-2}{\sqrt{5}} + \frac{6}{\sqrt{5}} = \frac{4}{\sqrt{5}}$$



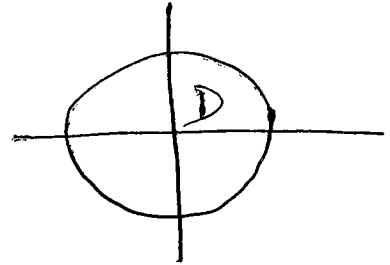
$$\boxed{\frac{4}{\sqrt{5}}}$$

3. (10 points) Find the absolute maximum and minimum of the function $f(x, y) = 4y^2 - x^2$ on the set $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$.

$$f_x = -2x, \quad f_y = 8y$$

critical points: $-2x = 0, \quad 8y = 0$

so $(0, 0)$



boundary:

on the circle, $x^2 = 1 - y^2$

So $f(x, y) = g(y) = 4y^2 - 1 + y^2 = 5y^2 - 1$

find extreme values on $-1 \leq y \leq 1$:

$$g'(y) = 10y = 0 \Rightarrow \underline{y = 0}$$

endpoints: $y = \pm 1$



On the circle, $y = 0 \Rightarrow x = \pm 1$

and $y = \pm 1 \Rightarrow x = 0$

So we look at $(0, 1), (0, -1), (1, 0), (-1, 0), (0, 0)$.

$$f(0, 0) = 0$$

$$f(0, 1) = 4$$

$$f(0, -1) = 4$$

$$f(1, 0) = -1$$

$$f(-1, 0) = -1$$

$$\text{absolute max} = 4$$

$$\text{absolute min} = -1$$

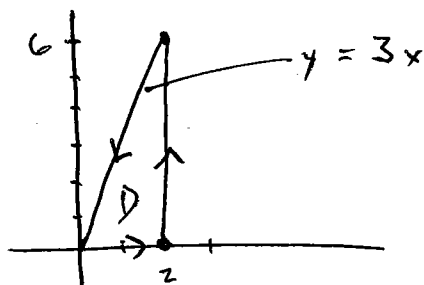
4. (8 points) Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = \langle y^2 \cos x, x^2 + 2y \sin x \rangle$ and C is the triangle from $(0, 0)$ to $(2, 0)$ to $(2, 6)$ to $(0, 0)$.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (Q_x - P_y) dA$$

$$= \iint_D 2x + 2y \cos x - 2y \cos x dA$$

$$= \int_0^2 \int_0^{3x} 2x dy dx = \int_0^2 2xy \Big|_0^{3x} dx$$

$$= \int_0^2 6x^2 dx = 2x^3 \Big|_0^2 = \boxed{16}$$



5. (8 points) Find the local maxima, local minima, and saddle points of the function $f(x, y) = x^2 + y^2 + \frac{1}{x^4 y^4}$.

$$f_x = 2x - 4x^{-5}y^{-4} = 0 \Rightarrow 2x = 4x^{-5}y^{-4}$$

$$\text{So } y^4 = 2x^{-6} \Rightarrow \textcircled{1} y = 2^{1/4} x^{-6/4}$$

$$\text{Similarly } f_y = 2y - 4x^{-4}y^{-5} = 0 \Rightarrow \dots \Rightarrow x = 2^{1/4} y^{-6/4} \textcircled{2}$$

$$\text{Put } \textcircled{1} \text{ into } \textcircled{2}: x = 2^{1/4} (2^{1/4} x^{-6/4})^{-6/4}$$

$$x = 2^{1/4 - 9/16} x^{36/16} \Rightarrow 2^{1/8} = x^{20/16} = x^{5/4}$$

$$\Rightarrow \underline{x = 2^{1/10}}$$

$$\text{by symmetry } \underline{y = 2^{1/10}}$$

$$f_{xx} = 2 + 20x^{-6}y^{-4}$$

$$f_{yy} = 2 + 20x^{-4}y^{-6}$$

$$f_{xy} = 16x^{-5}y^{-5}$$

$$\text{At } (2^{1/10}, 2^{1/10}) \text{ we have } f_{xx} = 12, f_{yy} = 12, f_{xy} = 8.$$

$$D(2^{1/10}, 2^{1/10}) = 144 - 64 = 80$$

$$D, f_{xx} > 0, \text{ so } \boxed{\text{local min. at } (2^{1/10}, 2^{1/10})}$$

6a. (2 points) Use spherical coordinates to write down a parametrization $\mathbf{r}(\phi, \theta)$ of the sphere of radius two, with domain $D = \{(\phi, \theta) \mid 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\}$.

$$\mathbf{r}(\phi, \theta) = \langle 2 \sin \phi \cos \theta, 2 \sin \phi \sin \theta, 2 \cos \phi \rangle$$

6b. (6 points) Find \mathbf{r}_ϕ , \mathbf{r}_θ , and $\mathbf{r}_\phi \times \mathbf{r}_\theta$. Then carry out the calculation that $dS = 4 \sin \phi dA$.

$$\mathbf{r}_\phi = \langle 2 \cos \phi \cos \theta, 2 \cos \phi \sin \theta, -2 \sin \phi \rangle$$

$$\mathbf{r}_\theta = \langle -2 \sin \phi \sin \theta, 2 \sin \phi \cos \theta, 0 \rangle$$

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = \langle 4 \sin^2 \phi \cos \theta, 4 \sin^2 \phi \sin \theta, \underbrace{4 \cos \phi \sin \phi \cos^2 \theta + 4 \sin \phi \cos \phi \sin^2 \theta}_{= 4 \cos \phi \sin \phi} \rangle$$

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = (16 \sin^4 \phi \cos^2 \theta + 16 \sin^4 \phi \sin^2 \theta + 16 \cos^2 \phi \sin^2 \phi)^{1/2}$$

$$= (16 \sin^4 \phi + 16 \cos^2 \phi \sin^2 \phi)^{1/2}$$

$$= (16 \sin^2 \phi \sin^2 \phi + 16 \cos^2 \phi \sin^2 \phi)^{1/2}$$

$$= (16 \sin^2 \phi)^{1/2} = 4 \sin \phi$$

$$dS = |\mathbf{r}_\phi \times \mathbf{r}_\theta| dA = 4 \sin \phi dA$$

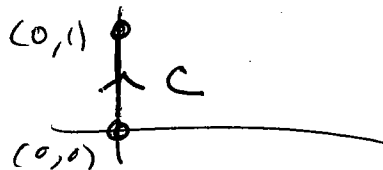
(continued)

6c. (6 points) Use the Divergence Theorem to evaluate $\iiint_E \operatorname{div} \mathbf{F} \, dV$ where E is the region enclosed by the sphere of radius 2 and $\mathbf{F}(x, y, z) = \langle y, -x, z \rangle$. You may use the fact that $\mathbf{n} = \langle \frac{1}{2}x, \frac{1}{2}y, \frac{1}{2}z \rangle$.

$$\begin{aligned} \iiint_E \operatorname{div} \mathbf{F} \, dV &= \iint_S \mathbf{F} \cdot \mathbf{n} \, dS \\ &= \iint_S \langle y, -x, z \rangle \cdot \left\langle \frac{x}{2}, \frac{y}{2}, \frac{z}{2} \right\rangle dS \\ &= \iint_S \frac{1}{2} z^2 dS = \iint_D \frac{1}{2} (2 \cos \varphi)^2 4 \sin \varphi \, dA \\ &= \int_0^{2\pi} \int_0^{\pi} 8 \cos^2 \varphi \sin \varphi \, d\varphi \, d\theta \quad \left[\begin{array}{l} u = \cos \varphi \\ du = -\sin \varphi \, d\varphi \end{array} \right] \\ &= \int_0^{2\pi} d\theta \int_1^{-1} -8u^2 \, du = 2\pi \left(-\frac{8}{3} u^3 \Big|_1^{-1} \right) \\ &= 2\pi \left(\frac{8}{3} + \frac{8}{3} \right) = \boxed{\frac{32\pi}{3}} \end{aligned}$$

7. (6 points) A certain function $f(x, y)$ has gradient $\langle e^{x^2}, ye^{y^2} \rangle$. Its value at $(0, 0)$ is 4. Find its value at $(0, 1)$. [Note: you cannot integrate e^{x^2} , so you cannot calculate an explicit expression for $f(x, y)$.]

$$\mathbf{r}(t) = \langle 0, t \rangle, \quad 0 \leq t \leq 1$$



Use F.T.L.I:

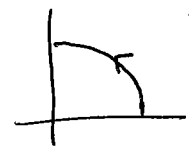
$$\int_C \nabla f \cdot d\mathbf{r} = f(0, 1) - f(0, 0)$$

$$\begin{aligned} \text{So } f(0, 1) &= \int_C \nabla f \cdot d\mathbf{r} + 4 & \mathbf{r}'(t) &= \langle 0, 1 \rangle \\ &= \int_0^1 \langle e^{0^2}, te^{t^2} \rangle \cdot \langle 0, 1 \rangle dt + 4 \\ &= \int_0^1 te^{t^2} dt + 4 = \frac{1}{2} e^{t^2} \Big|_0^1 + 4 \\ &= \frac{1}{2}(e-1) + 4 = \boxed{\frac{1}{2}(e+7)} \end{aligned}$$

8. (8 points) Let C be the portion of the unit circle $x^2 + y^2 = 1$ that lies in the first quadrant, starting at $(1, 0)$ and ending at $(0, 1)$.

(a) Write a parametrization $r(t)$ for C .

$$r(t) = \langle \cos t, \sin t \rangle, \quad 0 \leq t \leq \frac{\pi}{2}$$



Next express the following line integrals as definite integrals involving t . Simplify, but do not evaluate them.

(b) $\int_C x^4 y^2 dx$

$$dx = x'(t) dt$$

$$\int_0^{\pi/2} (\cos t)^4 (\sin t)^2 (-\sin t) dt$$

(c) $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = \langle 3xy, x + y \rangle$

$$r'(t) = \langle -\sin t, \cos t \rangle$$

$$\int_0^{\pi/2} \langle 3 \cos t \sin t, \cos t + \sin t \rangle \cdot \langle -\sin t, \cos t \rangle dt$$

$$= \int_0^{\pi/2} -3 \cos t \sin^2 t + \cos^2 t + \sin t \cos t dt$$

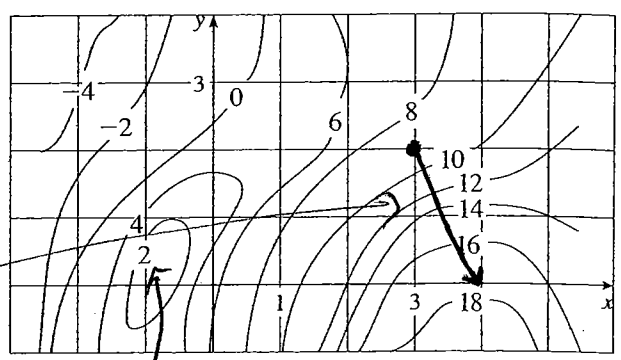
9. (8 points) A contour map is given for the function $f(x, y)$. Estimate $f_x(3, 2)$ and $f_y(3, 2)$. Then draw carefully the gradient vector at $(3, 2)$.

Are there any critical points in the region shown? If so, where?

$$f_x(3, 2) \approx \frac{2}{2} = 1$$

$$f_y(3, 2) \approx \frac{-2}{1} = -2$$

$$\nabla f(3, 2) = \langle 1, -2 \rangle$$



critical point near $(1, 0)$