1. (8 points) Let C be the portion of the unit circle $x^2 + y^2 = 1$ that lies in the first quadrant, starting at $(1, 0)$ and ending at $(0, 1)$.

(a) Write a parametrization $\mathbf{r}(t)$ for C.

$$\boxed{\underline{r}(t) = \langle \cos t, \sin t \rangle \qquad 0 \le t \le \pi/2}$$

Next express the following line integrals as definite integrals involving t. Simplify, but do not evaluate them.

(b) $\int_C (xy)^3\, dy$

$dy = y'(t)\, dt$

$$\boxed{\int_0^{\pi/2} (\cos t \sin t)^3 \cos t\, dt}$$

(c) $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = \langle 4xy, x - y \rangle$

$\underline{r}'(t) = \langle -\sin t, \cos t \rangle$

$$\int_0^{\pi/2} \langle 4\cos t \sin t, \cos t - \sin t \rangle \cdot \langle -\sin t, \cos t \rangle\, dt$$

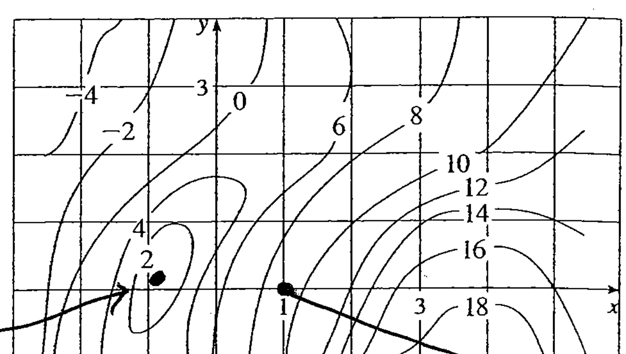
$$= \boxed{\int_0^{\pi/2} -4\cos t \sin^2 t + \cos^2 t - \sin t \cos t\, dt}$$

2. (8 points) A contour map is given for the function $f(x, y)$. Estimate $f_x(1, 0)$ and $f_y(1, 0)$. Then draw carefully the gradient vector at $(1, 0)$.

Are there any critical points in the region shown? If so, where?

$f_x(1,0) \approx \frac{2}{1/2} = 4$

$f_y(1,0) \approx \frac{-2}{7/4} = -8/7$



critical point near $(-1,0)$

$\nabla f(1,0) = \langle 4, -8/7 \rangle$

3a. (2 points) Use spherical coordinates to write down a parametrization $\mathbf{r}(\phi, \theta)$ of the sphere of radius *two*, with domain $D = \{(\phi, \theta) \mid 0 \le \phi \le \pi,\ \ 0 \le \theta \le 2\pi\}$.

$$\boxed{\mathbf{r}(\varphi,\theta) = \langle 2\sin\varphi\cos\theta,\ 2\sin\varphi\sin\theta,\ 2\cos\varphi \rangle}$$

3b. (6 points) Find $\mathbf{r}_\phi$, $\mathbf{r}_\theta$, and $\mathbf{r}_\phi \times \mathbf{r}_\theta$. Then show that dS is $4\sin\phi\, dA$.

$$\mathbf{r}_\varphi = \langle 2\cos\varphi\cos\theta,\ 2\cos\varphi\sin\theta,\ -2\sin\varphi \rangle$$

$$\mathbf{r}_\theta = \langle -2\sin\varphi\sin\theta,\ 2\sin\varphi\cos\theta,\ 0 \rangle$$

$$\mathbf{r}_\varphi \times \mathbf{r}_\theta = \langle 4\sin^2\varphi\cos\theta,\ 4\sin^2\varphi\sin\theta,\ \underbrace{4\cos\varphi\sin\varphi\cos^2\theta + 4\sin\varphi\cos\varphi\sin^2\theta}_{=4\cos\varphi\sin\varphi} \rangle$$

$$|\mathbf{r}_\varphi \times \mathbf{r}_\theta| = (16\sin^4\varphi\cos^2\theta + 16\sin^4\varphi\sin^2\theta + 16\cos^2\varphi\sin^2\varphi)^{1/2}$$

$$= (16\sin^4\varphi + 16\cos^2\varphi\sin^2\varphi)^{1/2}$$

$$= (16\sin^2\varphi\sin^2\varphi + 16\cos^2\varphi\sin^2\varphi)^{1/2}$$

$$= (16\sin^2\varphi)^{1/2} = 4\sin\varphi.$$

$$dS = |\mathbf{r}_\varphi \times \mathbf{r}_\theta|\, dA = 4\sin\varphi\, dA$$

(continued)

3c. (6 points) Use the Divergence Theorem to evaluate $\iiint_E \operatorname{div}\mathbf{F}\, dV$ where E is the region enclosed by the sphere of radius 2 and $\mathbf{F}(x,y,z) = \langle y, -x, z\rangle$. You may use the fact that $\mathbf{n} = \langle \frac{1}{2}x, \frac{1}{2}y, \frac{1}{2}z\rangle$.

$$\iiint_E \operatorname{div} F\, dV = \iint_S F \cdot n\, dS$$

$$= \iint_S \langle y, -x, z\rangle \cdot \langle \tfrac{x}{2}, \tfrac{y}{2}, \tfrac{z}{2}\rangle\, dS$$

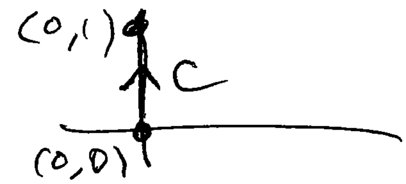
$$= \iint_S \tfrac{1}{2}z^2\, dS = \iint_D \tfrac{1}{2}(2\cos\varphi)^2 4\sin\varphi\, dA$$

$$= \int_0^{2\pi}\int_0^{\pi} 8\cos^2\varphi \sin\varphi\, d\varphi\, d\theta \quad \left[\begin{array}{l} u = \cos\varphi \\ du = -\sin\varphi\, d\varphi \end{array}\right]$$

$$= \int_0^{2\pi} d\theta \int_1^{-1} -8u^2\, du = 2\pi\left(-\tfrac{8}{3}u^3 \Big|_1^{-1}\right)$$

$$= 2\pi\left(\tfrac{8}{3} + \tfrac{8}{3}\right) = \boxed{\frac{32\pi}{3}}$$

4. (6 points) A certain function $f(x,y)$ has gradient $\langle e^{x^2}, y\sin(y^2)\rangle$. Its value at $(0,0)$ is 4. Find its value at $(0,1)$. [Note: you cannot integrate e^{x^2}, so you cannot calculate an explicit expression for $f(x,y)$.]



$r(t) = \langle 0, t\rangle, \quad 0 \le t \le 1$

USE F.T.L.I.

$$\int_C \nabla f \cdot dr = f(0,1) - f(0,0)$$

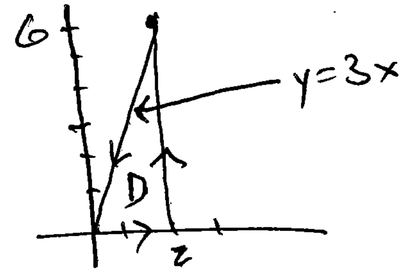
So $f(0,1) = \int_C \nabla f \cdot dr + 4 \qquad r'(t) = \langle 0, 1\rangle$

$$= \int_0^1 \langle e^{0^2}, t\sin(t^2)\rangle \cdot \langle 0, 1\rangle\, dt + 4$$

$$= \int_0^1 t\sin(t^2)\, dt + 4 = -\tfrac{1}{2}\cos(t^2)\Big|_0^1 + 4$$

$$= -\tfrac{1}{2}(\cos(1) - \cos(0)) + 4 = \boxed{\tfrac{1}{2}(9 - \cos(1))}$$

5. (8 points) Use Green's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = \langle y^2 \cos x, x^2 + 2y \sin x \rangle$ and C is the triangle from $(0,0)$ to $(2,0)$ to $(2,6)$ to $(0,0)$.



$$\int_C \underline{F} \cdot d\underline{r} = \iint_D (Q_x - P_y)\, dA$$

$$= \iint_D 2x + 2y\cos x - 2y\cos x \, dA$$

$$= \int_0^2 \int_0^{3x} 2x \, dy\, dx = \int_0^2 2xy \Big|_0^{3x} dx$$

$$= \int_0^2 6x^2 dx = 2x^3 \Big|_0^2 = \boxed{16}.$$

6. (8 points) Find the local maxima, local minima, and saddle points of the function $f(x, y) = x^2 + y^2 + \frac{1}{x^4 y^4}$.

$f_x = 2x - 4x^{-5}y^{-4} = 0 \Rightarrow 2x = 4x^{-5}y^{-4}$

so $y^4 = 2x^{-6} \Rightarrow$ ① $\underline{y = 2^{1/4} x^{-6/4}}$.

Similarly $f_y = 2y - 4x^{-4}y^{-5} = 0 \Rightarrow \cdots \Rightarrow$ ② $\underline{x = 2^{1/4} y^{-6/4}}$.

<u>Put ① into ②:</u> $x = 2^{1/4}(2^{1/4} x^{-6/4})^{-6/4}$

$x = 2^{1/4 - 6/16} x^{36/16} \Rightarrow 2^{1/8} = x^{20/16} = x^{5/4}$

$\Rightarrow \underline{x = 2^{1/10}}$.

by symmetry, $\underline{y = 2^{1/10}}$.

$f_{xx} = 2 + 20x^{-6}y^{-4}$

$f_{yy} = 2 + 20x^{-4}y^{-6}$

$f_{xy} = 16x^{-5}y^{-5}$

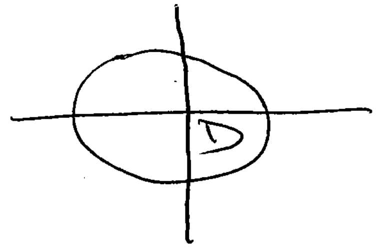
At $(2^{1/10}, 2^{1/10})$ we have $f_{xx} = 12$, $f_{yy} = 12$, $f_{xy} = 8$.

$D(2^{1/10}, 2^{1/10}) = 144 - 64 = 80$.

$D, f_{xx} > 0$, so $\boxed{\text{local min. at } (2^{1/10}, 2^{1/10})}$

7. (10 points) Find the absolute maximum and minimum of the function $f(x,y) = 4y^2 - x^2$ on the set $D = \{(x,y) \mid x^2 + y^2 \leq 1\}$.

$f_x = -2x$, $f_y = 8y$



critical points : $-2x = 0$, $8y = 0$

So $(0,0)$.

boundary :

On the circle, $x^2 = 1 - y^2$

So $f(x,y) = g(y) = 4y^2 - 1 + y^2 = 5y^2 - 1$.

find extreme values on $-1 \leq y \leq 1$:

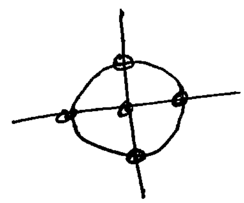
$g'(y) = 10y = 0$

$\Rightarrow y = 0$

also endpoints: $y = \pm 1$.

On the circle, $y = 0 \Rightarrow x = \pm 1$

and $y = \pm 1 \Rightarrow x = 0$



So we look at $(0,0), (1,0), (-1,0), (0,1), (0,-1)$:

$f(0,0) = 0$

$f(1,0) = -1$

$f(-1,0) = -1$

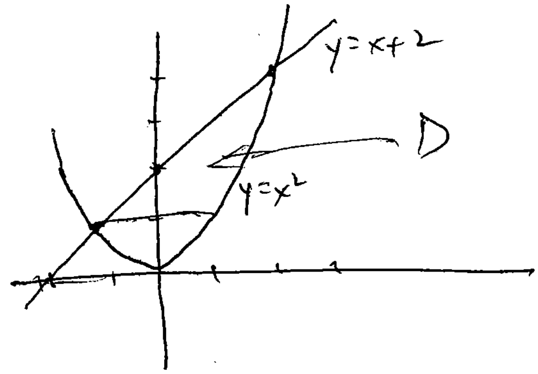
$f(0,1) = 4$

$f(0,-1) = 4$

absolute max $= 4$

absolute min $= -1$

8. (8 points) Let D be the region in the xy–plane between the curves $y = x^2$ and $y = x + 2$. Draw the region D carefully. Then set up the integral $\iint_D f(x, y)\,dA$ in two different ways, using $dA = dx\,dy$ and $dA = dy\,dx$.

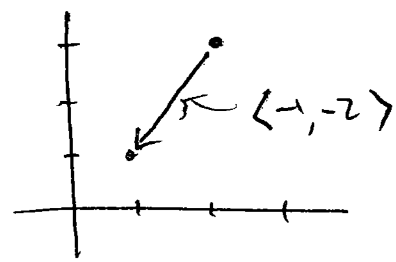


intersections:

$$x^2 = x + 2$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$
$$x = 2, -1$$

$$\boxed{\begin{gathered}\int_{-1}^{2}\int_{x^2}^{x+2} f(x,y)\,dy\,dx \\ \int_0^1\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\,dx\,dy + \int_1^4\int_{y-2}^{\sqrt{y}} f(x,y)\,dx\,dy\end{gathered}}$$

9. (8 points) Use the gradient to find the directional derivative of $f(x, y) = x^3y^4 - 3x^2$ at the point $(2, 3)$ in the direction of the point $(1, 1)$.



$$\nabla f = \langle 3x^2y^4 - 6x,\ 4x^3y^3 \rangle$$

$$\nabla f(2,3) = \langle 3\cdot 4\cdot 81 - 12,\ 4\cdot 8\cdot 27 \rangle$$
$$= \langle 960, 864 \rangle$$

$$\underline{u} = \frac{\langle -1,-2\rangle}{|\langle -1,-2\rangle|} = \left\langle \frac{-1}{\sqrt{5}}, \frac{-2}{\sqrt{5}} \right\rangle.$$

$$D_u f(2,3) = \nabla f(2,3)\cdot \underline{u} = \frac{-960}{\sqrt{5}} - \frac{2\cdot 864}{\sqrt{5}} = \boxed{\frac{-2738}{\sqrt{5}}}$$