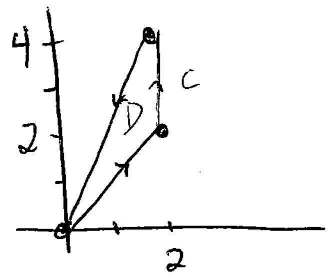
1(a) Use Green's Theorem to evaluate $\int_C xy^2\,dx + 2x^2y\,dy$ where C is the triangle with vertices $(0,0)$, $(2,2)$, and $(2,4)$, oriented positively.



$P(x,y) = xy^2$, $Q(x,y) = 2x^2y$

Green's Theorem: $\int_C P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dA$

$$= \iint_D (4xy - 2xy)\,dA = \int_0^2 \int_x^{2x} 2xy\,dy\,dx$$

$$= \int_0^2 xy^2 \Big|_x^{2x} dx = \int_0^2 4x^3 - x^3\,dx = \int_0^2 3x^3\,dx$$

$$= \frac{3}{4}x^4 \Big|_0^2 = \frac{3}{4}(16 - 0) = \boxed{12}$$

1(b) Give a vector field $\mathbf{F}(x,y)$ with the property that for any positively oriented simple closed curve C, the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ measures the area enclosed by C. Why does $\mathbf{F}$ have this property?

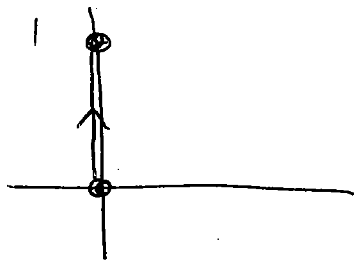
Any $\underline{F}$ with $Q_x - P_y = 1$ works, since by Green's Theorem, $\int_C \underline{F}\cdot d\underline{r} = \iint_D (Q_x - P_y)\,dA$

$= \iint_D 1\,dA = \text{Area}(D)$.

$\boxed{F(x,y) = \langle 0, x\rangle}$ works

(also $\langle -y, 0\rangle$, $\frac{1}{2}\langle -y, x\rangle$, etc)

2(a) A certain function $f(x,y)$ has gradient $\langle e^{x^2}, ye^{y^2}\rangle$. Its value at $(0,0)$ is 5. Use a line integral to find $f(0,1)$. [Note: the function e^{x^2} cannot be integrated explicitly, so you will not be able to find a formula for f.]



use fund. theorem!

$$\int_C \nabla f \cdot d\underline{r} = f(\underline{r}(b)) - f(\underline{r}(a))$$

$$\underline{r}(t) = \langle 0, t\rangle, \quad 0 \le t \le 1$$

$$\int_0^1 \langle e^{x^2}, ye^{y^2}\rangle \cdot \langle 0,1\rangle \, dt = \int_0^1 te^{t^2} dt$$

$$= \frac{1}{2} e^{t^2}\Big|_0^1 = \frac{1}{2}(e-1). \quad \text{So } \frac{1}{2}(e-1) = f(0,1) - 5$$

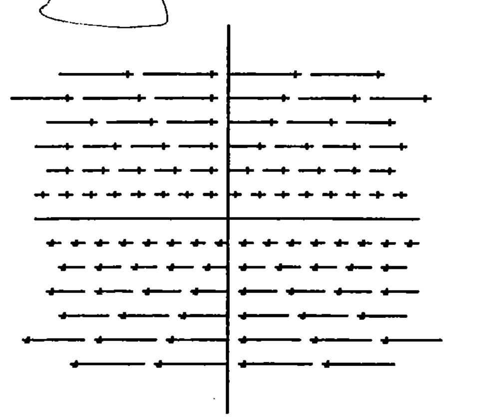
$$f(0,1) = \boxed{\frac{1}{2}(e-1) + 5}$$

2(b) The figure below shows a vector field $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$.
For each of the following quantities, say whether it is positive, negative, or zero:

- $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the line segment from $(1,0)$ to $(1,2)$ $\boxed{0}$
- $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the line segment from $(0,1)$ to $(2,1)$ $\boxed{>0}$
- $\int_C \mathbf{F} \cdot d\mathbf{r}$ where C is the unit circle, oriented counter clockwise $\boxed{<0}$
- $\frac{\partial P}{\partial x}$ $\boxed{0}$
- $\frac{\partial P}{\partial y}$ $\boxed{>0}$
- $\frac{\partial Q}{\partial y}$ $\boxed{0}$

Finally, is $\mathbf{F}$ conservative? Say briefly why or why not.

$\boxed{F \text{ is not conservative}}$, since integrating around a closed curve does not give zero.



3(a) A curve C is parametrized as $\mathbf{r}(t) = \langle \cos t, 2e^t \rangle$ for $0 \le t \le \pi$. Express each of the following as ordinary integrals in t. *Do not* evaluate the integrals.

- $\int_C x^2 y \, ds$
- $\int_C x^2 y \, dx$
- $\int_C \langle x^2, y \rangle \cdot d\mathbf{r}$

$r'(t) = \langle -\sin t, 2e^t \rangle$

$ds = \sqrt{\sin^2 t + 4e^{2t}} \, dt$

$dx = -\sin t \, dt$

$$\boxed{\begin{aligned} \int_C x^2 y \, ds &= \int_0^\pi \cos^2 t \cdot 2e^t \sqrt{\sin^2 t + 4e^{2t}} \, dt \\ \int_C x^2 y \, dx &= \int_0^\pi \cos^2 t \cdot 2e^t(-\sin t) \, dt \\ \int_C \langle x^2, y \rangle \cdot d\underline{r} &= \int_0^\pi -\cos^2 t \sin t + 4e^{2t} \, dt \end{aligned}}$$

(b) Describe carefully in words the following surfaces (given with respect to spherical coordinates):

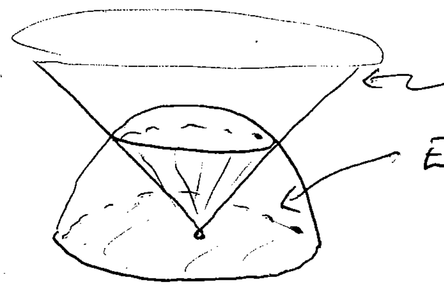
- $\rho = 3$
- $\theta = \pi/2$
- $\phi = \pi/2$

- sphere of radius 3 centered at origin
- half-plane with boundary the z-axis and containing the positive y-axis
- the xy-plane

4. Let E be the solid region that lies within the sphere $x^2+y^2+z^2=4$, above the xy-plane, and below the cone $z=\sqrt{x^2+y^2}$.

(a) Draw the region E and describe it using spherical coordinates.

(b) Find the volume of E.



cone $z=r$, so angle $\pi/4$ from vertical

$$E=\{(\rho,\theta,\varphi) \mid 0\le\rho\le 2,\ 0\le\theta\le 2\pi,\ \pi/4\le\varphi\le\pi/2\}$$

$$\mathrm{Vol}(E)=\iiint_E 1\,dV$$

$$=\int_{\pi/4}^{\pi/2}\int_0^{2\pi}\int_0^2 \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

$$=\int_0^2\rho^2\,d\rho\cdot\int_0^{2\pi}d\theta\cdot\int_{\pi/4}^{\pi/2}\sin\varphi\,d\varphi$$

$$=\left(\tfrac{1}{3}\rho^3\Big|_0^2\right)(2\pi)\left(-\cos\varphi\Big|_{\pi/4}^{\pi/2}\right)$$

$$=\left(\tfrac{8}{3}\right)(2\pi)\left(\tfrac{\sqrt{2}}{2}\right)=\boxed{\tfrac{8\sqrt{2}}{3}\pi}$$