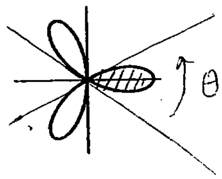
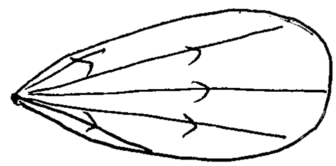
1(a) The figure below shows the graph of the polar equation $r = \cos(3\theta)$. Use a double integral in polar coordinates to find the area enclosed by one of its loops. [You may need the identity $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$.]





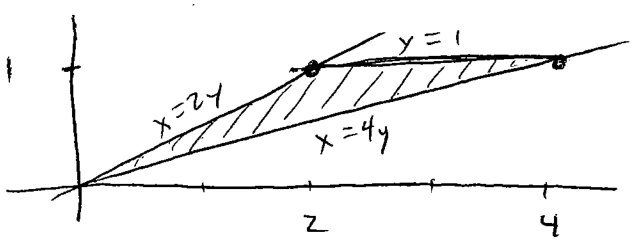
$\cos(3\theta) = 0$ when $3\theta = \pm \pi/2$, or $\theta = \pm \pi/6$,

$$\text{Area} = \int_{-\pi/6}^{\pi/6} \int_0^{\cos(3\theta)} r \, dr \, d\theta = \int_{-\pi/6}^{\pi/6} \frac{1}{2} r^2 \Big|_0^{\cos(3\theta)} d\theta = \int_{-\pi/6}^{\pi/6} \frac{1}{2} \cos^2(3\theta) \, d\theta$$

$$= \int_{-\pi/6}^{\pi/6} \frac{1}{4}\left(1 + \cos(6\theta)\right) d\theta = \frac{1}{4}\left[\theta + \frac{1}{6}\sin(6\theta)\right] \Big|_{-\pi/6}^{\pi/6}$$

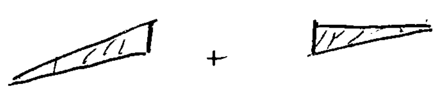
$$= \frac{1}{4}\left(\frac{\pi}{6} + \frac{\pi}{6}\right) = \boxed{\frac{\pi}{12}}$$

1(b) For the integral $\int_0^1 \int_{2y}^{4y} f(x,y) \, dx \, dy$, sketch the region of integration and change the order of integration. The answer should have two terms.



← region

Split into two:



$$\boxed{\int_0^2 \int_{x/4}^{x/2} f(x,y) \, dy \, dx + \int_2^4 \int_{x/4}^{1} f(x,y) \, dy \, dx}$$

2(a) Suppose we want to find the point on the surface $xy\sin(z) = 4$ that is closest to the point $(1,1,1)$. Write down a system of equations whose solution will include this closest point. [Do not solve the system or proceed any further.]

Let $f(x,y,z)$ be squared distance from (x,y,z) to $(1,1,1)$.

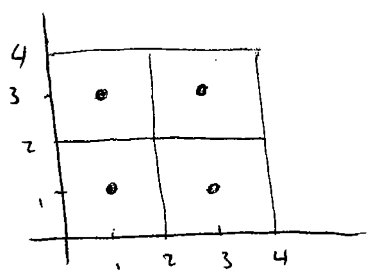
So $f(x,y,z) = (x-1)^2 + (y-1)^2 + (z-1)^2$, Want to minimize f subject to constraint $\underbrace{xy\sin(z)}_{g(x,y,z)} = 4$.

Lagrange multipliers: $\nabla f = \langle 2(x-1), 2(y-1), 2(z-1) \rangle$

System: $\nabla g = \langle y\sin z, x\sin z, xy\cos z \rangle$

$$\boxed{\begin{aligned} \lambda y\sin z &= 2(x-1) \\ \lambda x\sin z &= 2(y-1) \\ \lambda xy\cos z &= 2(z-1) \\ xy\sin z &= 4 \end{aligned}}$$

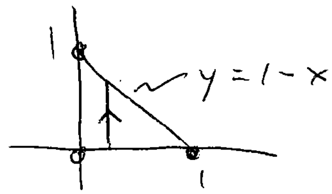
2(b) If $R = [0,4] \times [0,4]$, use a Riemann sum with $m = 2$, $n = 2$ to write down a numerical expression estimating $\iint_R x^2 - y^2 \, dA$. Use the Midpoint Rule for your sample points. [A complete expression, with numbers only, is required; you do not need to do the arithmetic to get the final number.]



$\Delta A = 4$

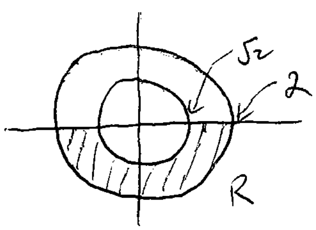
Riemann Sum $= \boxed{(1^2 - 1^2)\cdot 4 + (1^2 - 3^2)\cdot 4 + (3^2 - 1^2)\cdot 4 + (3^2 - 3^2)\cdot 4}$

3(a) A triangular lamina with vertices $(0,0)$, $(1,0)$, and $(0,1)$ has density function $\rho(x,y) = 2(x+y)$. Write down explicit formulas (involving integrals) for the mass, and for $\bar{x}$ and $\bar{y}$, the coordinates of the center of mass. Do not evaluate the integrals.



$$\boxed{\begin{aligned} m &= \int_0^1 \int_0^{1-x} 2(x+y)\,dy\,dx \\ \bar{x} &= \frac{1}{m}\int_0^1 \int_0^{1-x} 2x(x+y)\,dy\,dx \\ \bar{y} &= \frac{1}{m}\int_0^1 \int_0^{1-x} 2y(x+y)\,dy\,dx \end{aligned}}$$

(b) Evaluate by changing to polar coordinates: $\iint_R (x+y)\,dA$, where R is the region that lies below the x-axis and between the circles $x^2+y^2=2$ and $x^2+y^2=4$.



$$\int_{\pi}^{2\pi}\int_{\sqrt{2}}^{2} (r\cos\theta + r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_{\pi}^{2\pi} \int_{\sqrt{2}}^{2} r^2\,dr\,(\cos\theta + \sin\theta)\,d\theta$$

$$= \int_{\pi}^{2\pi} \frac{1}{3} r^3 \Big|_{\sqrt{2}}^{2} (\cos\theta + \sin\theta)\,d\theta$$

$$= \frac{8-2\sqrt{2}}{3}\int_{\pi}^{2\pi} \cos\theta + \sin\theta\,d\theta = \frac{8-2\sqrt{2}}{3}\left(\sin\theta - \cos\theta \Big|_{\pi}^{2\pi}\right)$$

$$= \frac{8-2\sqrt{2}}{3}(-1-1) = \boxed{\frac{-16+4\sqrt{2}}{3}}$$

4(a) Use cylindrical coordinates to evaluate $\iiint_E \sqrt{x^2+y^2}\,dV$, where E is the region that lies inside the cylinder $x^2+y^2=16$ and between the planes $z=-5$ and $z=4$.

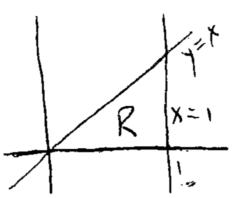
$$\iiint_E \sqrt{x^2+y^2}\,dV = \int_0^{2\pi}\int_0^4\int_{-5}^4 \underbrace{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}_{=r}\; r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^4 r^2 z \Big|_{-5}^4 dr\,d\theta = \int_0^{2\pi}\int_0^4 9r^2\,dr\,d\theta$$

$$= \int_0^{2\pi} 3r^3 \Big|_0^4 d\theta = \int_0^{2\pi} 3\cdot 64\,d\theta$$

$$= 2\pi\cdot 3\cdot 64 = \boxed{384\pi}$$

(b) Evaluate the integral $\iint_R e^{x^2}\,dA$, where R is the region bounded by the lines $y=0$, $x=y$, and $x=1$. [Warning: you must choose the correct order of integration for it to work.]



$$\int_0^1\int_0^x e^{x^2}\,dy\,dx = \int_0^1 e^{x^2} y \Big|_0^x dx$$

$$= \int_0^1 x e^{x^2}\,dx \qquad u=x^2,\ du = 2x\,dx$$

$$= \int_0^1 \tfrac{1}{2} e^u\,du = \tfrac{1}{2} e^u \Big|_0^1$$

$$= \boxed{\tfrac{1}{2}(e-1)}$$