1(a) For the function $f(t,u,v) = t^2uv^3$, calculate ∇f, and use it to find the rate of change of f at $(1,1,1)$ in the direction toward $(3,1,-1)$.

$$\nabla f = \langle 2tuv^3, t^2v^3, 3t^2uv^2 \rangle$$

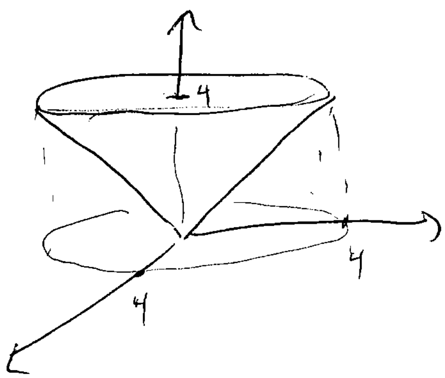
$$\nabla f(1,1,1) = \langle 2, 1, 3 \rangle$$

direction $= \langle 2, 0, -2 \rangle$, $\underline{u} = \langle \frac{2}{\sqrt{8}}, 0, \frac{-2}{\sqrt{8}} \rangle$

$$D_{\underline{u}} f(1,1,1) = \langle 2, 1, 3 \rangle \cdot \langle \frac{2}{\sqrt{8}}, 0, \frac{-2}{\sqrt{8}} \rangle$$

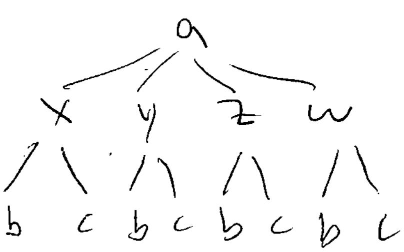
$$= \frac{4}{\sqrt{8}} - \frac{6}{\sqrt{8}} = \frac{-2}{\sqrt{8}} = \boxed{\frac{-1}{\sqrt{2}}}$$

(b) Sketch and describe in words the solid whose volume is given by the integral $\int_0^4 \int_0^{2\pi} \int_r^4 r\, dz\, d\theta\, dr$.



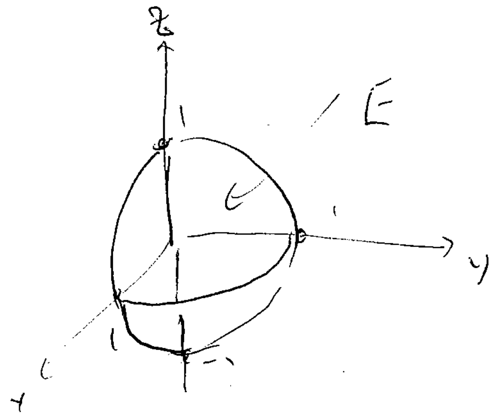
The solid cone of angle $\pi/4$ from the vertical, and height 4.

(c) Write down the chain rule for $\frac{\partial a}{\partial b}$ if $a = a(x,y,z,w)$, $x = x(b,c)$, $y = y(b,c)$, $z = z(b,c)$, and $w = w(b,c)$.



$$\boxed{\frac{\partial a}{\partial b} = \frac{\partial a}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial a}{\partial y}\frac{\partial y}{\partial b} + \frac{\partial a}{\partial z}\frac{\partial z}{\partial b} + \frac{\partial a}{\partial w}\frac{\partial w}{\partial b}}$$

2. Let E be the portion of the unit ball having positive x- and y-coordinates. (That is, E is given by $x^2+y^2+z^2 \le 1$, $x \ge 0$, $y \ge 0$.) Let S be the boundary surface of E, with outward normal $\mathbf{n}$. Use the Divergence Theorem to calculate $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$, where $\mathbf{F}(x,y,z) = \langle x^3, y^3, z^3\rangle$.



Divergence Theorem

$$\iint_S \underline{F}\cdot\underline{n}\,dS = \iiint_E \operatorname{div}\underline{F}\,dV$$

$$\operatorname{div}\underline{F} = 3x^2+3y^2+3z^2$$

$$\iiint_E 3(x^2+y^2+z^2)\,dV$$

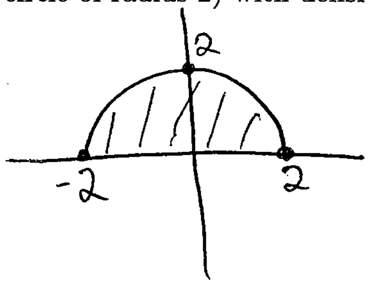
$$= \int_0^{\pi/2}\int_0^{\pi}\int_0^1 3\rho^2\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$= 3\int_0^1 \rho^4\,d\rho \int_0^{\pi}\sin\varphi\,d\varphi \int_0^{\pi/2} d\theta$$

$$= 3\left[\tfrac{1}{5}\rho^5\Big|_0^1\right]\left[-\cos\varphi\Big|_0^{\pi}\right]\cdot \pi/2$$

$$= 3\cdot\tfrac{1}{5}\cdot 2\cdot \pi/2 = \boxed{\frac{3\pi}{5}}$$

3 Find the mass and center of mass of the lamina shown below (above the x-axis and inside the circle of radius 2) with density given by $\rho(x,y) = \sqrt{x^2+y^2}$.



$$m = \iint_D \rho(x,y)\,dA = \int_0^{\pi}\int_0^2 r \cdot r\,dr\,d\theta$$

$$= \int_0^{\pi} \tfrac{1}{3}r^3\Big|_0^2 \,d\theta = \int_0^{\pi} \tfrac{8}{3}\,d\theta = \boxed{\tfrac{8\pi}{3}.}$$

$$\bar{x} = \frac{1}{m}\int_0^{\pi}\int_0^2 (r\cos\theta)\,r \cdot r\,dr\,d\theta = \frac{1}{m}\int_0^{\pi}\cos\theta\,d\theta \int_0^2 r^3\,dr$$

$$= \frac{1}{m}\left[\sin\theta\Big|_0^{\pi}\right]\left[\tfrac{1}{4}r^4\Big|_0^2\right]$$

$$= 0$$

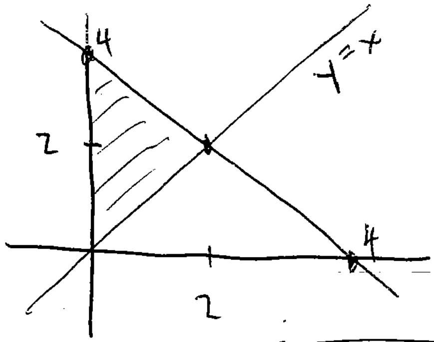
$$\bar{y} = \frac{1}{m}\int_0^{\pi}\int_0^2 (r\sin\theta)\,r \cdot r\,dr\,d\theta = \frac{1}{m}\int_0^{\pi}\sin\theta\,d\theta \int_0^2 r^3\,dr$$

$$= \frac{1}{m}\left[-\cos\theta\Big|_0^{\pi}\right]\left[\tfrac{1}{4}r^4\Big|_0^2\right]$$

$$= \frac{3}{8\pi}\cdot 2 \cdot 4 = \frac{3}{\pi}$$

center of mass $= \boxed{\left(0, \tfrac{3}{\pi}\right)}$

4(a) Let D be the region bounded by the lines $x = 0$, $x + y = 4$, and $y = x$. Set up the integral $\iint_D f(x, y)\, dA$ in two different ways, using $dA = dx\, dy$ and $dA = dy\, dx$.



$$\boxed{\int_0^2 \int_x^{4-x} f(x,y)\, dy\, dx}$$

and

$$\boxed{\int_0^2 \int_0^{y} f(x,y)\, dx\, dy + \int_2^4 \int_0^{4-y} f(x,y)\, dx\, dy}$$

(b) Find the critical points of $f(x, y) = 3x^2 - 2xy + y^2 - 8y$ and describe their local behavior, using the second derivative test.

$$f_x = 6x - 2y = 0$$
$$f_y = -2x + 2y - 8 = 0$$

$$4x \quad -8 = 0 \quad \Rightarrow \quad x = 2$$

$$6(2) - 2y = 0 \quad \Rightarrow \quad y = 6.$$

$\boxed{\text{Crit. Pt.: } (2,6)}$ $\qquad f_{xx} = 6, \; f_{yy} = 2, \; f_{xy} = -2$

$$D(2,6) = (6)(2) - (-2)^2 = 12 - 4 = 8 > 0$$

$$\rightarrow \quad \boxed{\text{local min. at } (2,6)}$$

5. Let S be the surface given by $x = u\cos(v)$, $y = u\sin(v)$, and $z = u$, where $0 \le u \le 1$ and $0 \le v \le 2\pi$.

(a) Carry out the steps in the computation that $|\mathbf{r}_u \times \mathbf{r}_v| = \sqrt{2}u$.

$$\underline{r}_u = \langle \cos v,\ \sin v,\ 1 \rangle$$

$$\underline{r}_v = \langle -u\sin v,\ u\cos v,\ 0 \rangle$$

$$\underline{r}_u \times \underline{r}_v = \langle -u\cos v,\ -u\sin v,\ \overbrace{u\cos^2 v + u\sin^2 v}^{=u} \rangle$$

$$|\underline{r}_u \times \underline{r}_v| = \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2} = \sqrt{2u^2} = \sqrt{2}\,u$$

(b) Calculate $\iint_S x^2\, dS$.

$$= \int_0^{2\pi}\int_0^1 u^2\cos^2 v \cdot \sqrt{2}\,u \, du\, dv$$

$$= \int_0^{2\pi} \cos^2 v \, dv \int_0^1 \sqrt{2}\,u^3 \, du$$

$$= \frac{1}{2}\int_0^{2\pi} (1 + \underbrace{\cos(2v)}_{\text{zero over } [0,2\pi]})\, dv \left(\frac{\sqrt{2}}{4} u^4 \Big|_0^1 \right)$$

$$= \left(\frac{1}{2}\right)(2\pi)\left(\frac{\sqrt{2}}{4}\right) = \boxed{\frac{\sqrt{2}\,\pi}{4}}$$

(c) Calculate $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F}(x, y, z) = \langle 0, 0, z^2 \rangle$.

$$\underline{n} = \frac{1}{\sqrt{2}u} \langle -u\cos v,\ -u\sin v,\ u \rangle$$

$$\underline{F} \cdot \underline{n} = \langle 0, 0, u^2 \rangle \cdot \frac{1}{\sqrt{2}u} \langle -u\cos v, -u\sin v, u \rangle$$

$$= \frac{1}{\sqrt{2}u} u^3$$

$$\iint_S \underline{F} \cdot \underline{n}\, dS = \int_0^{2\pi}\int_0^1 u^3 \, du\, dv = \int_0^{2\pi} \frac{1}{4}\, dv = \boxed{\frac{\pi}{2}}$$

6(a) Is there a vector field $\mathbf{F}$ such that $\operatorname{curl}\mathbf{F} = \langle x\sin y, \cos y, z - xy\rangle$? If so, find $\mathbf{F}$, and if not, say why not.

$$\operatorname{div}\langle x\sin y, \cos y, z-xy\rangle = \sin y - \sin y + 1 = 1 \quad \boxed{\text{NO}}$$

since divergence is not zero, this vector field cannot be the curl of anything (because $\operatorname{div}(\operatorname{curl} \underline{F}) = 0$).

(b) Is the vector field $\mathbf{F}(x,y,z) = \langle 2xy, x^2+2yz, y^2\rangle$ conservative? If so, find f such that $\nabla f = \mathbf{F}$, and if not, say why not.

$$f_x = 2xy \Rightarrow f(x,y,z) = x^2y + C(y,z)$$

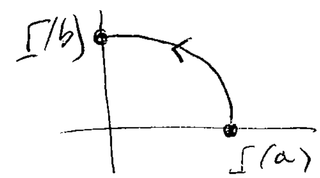
$$f_y = x^2 + 2yz \Rightarrow f = x^2y + y^2z + D(x,z)$$

$$f_z = y^2 \Rightarrow f = y^2z + E(x,y)$$

$$\boxed{f(x,y,z) = x^2y + y^2z + K}$$

has gradient $\underline{F}$, so $\boxed{\underline{F} \text{ is conservative.}}$

(c) Calculate $\int_C \nabla f \cdot d\mathbf{r}$ where $f(x,y) = \sqrt{\sin(x^2) + \cos(xy)}$ and C is the portion of the unit circle in the xy-plane from $(1,0)$ to $(0,1)$. [Think.]



$$\int_C \nabla f \cdot d\underline{r} = f(\underline{r}(b)) - f(\underline{r}(a))$$

$$= f(0,1) - f(1,0)$$

$$= \boxed{1 - \sqrt{\sin(1)+1}}$$