

1(a) Find a formula for the n^{th} term of the sequence $(\frac{1}{2}, -\frac{4}{3}, \frac{9}{4}, -\frac{16}{5}, \frac{25}{6}, \dots)$.

$$a_n = (-1)^{n+1} \frac{n^2}{n+1}$$

(b) Consider the sequence $a_n = \frac{n \sin n}{n^2 + 1}$. Does it converge, and if so, what is the limit? Explain your reasoning.

$$-1 \leq \sin n \leq 1, \text{ so } \frac{-n}{n^2+1} \leq \frac{n \sin n}{n^2+1} \leq \frac{n}{n^2+1}.$$

Also, $\frac{n}{n^2+1} \rightarrow 0$ because $\lim_{x \rightarrow \infty} \frac{x}{x^2+1} = \lim_{x \rightarrow \infty} \frac{\sqrt{x}}{1+\sqrt{x^2}} = \frac{0}{1} = 0$

and $\frac{-n}{n^2+1} \rightarrow -0 = 0$ (limit law)

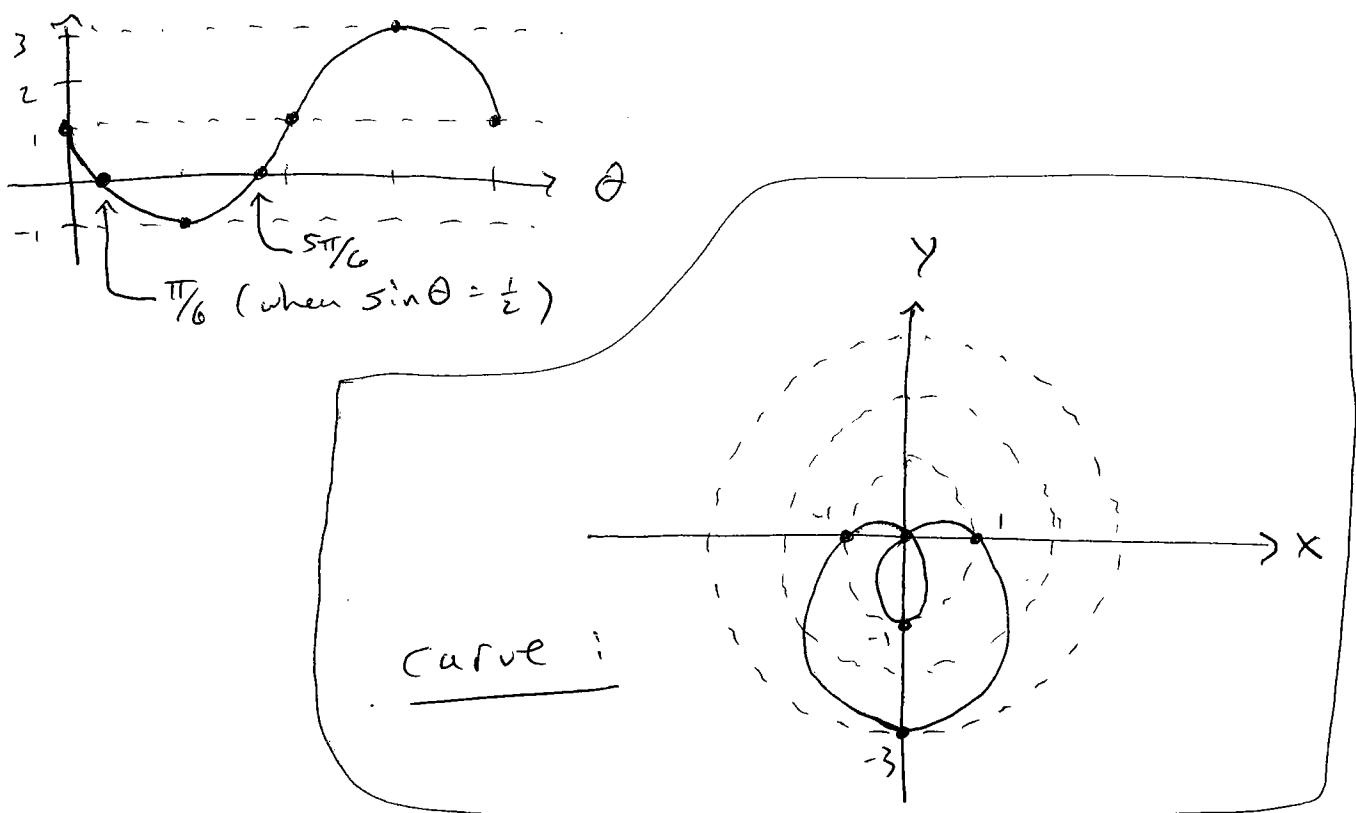
By Squeeze Theorem, $\frac{n \sin n}{n^2+1} \rightarrow 0$.

(c) Say carefully what it means for a sequence $\{a_n\}$ to be *monotone*.

It is monotone if
either $a_{n+1} > a_n$ for all n

or $a_{n+1} < a_n$ for all n

2(a) Sketch carefully the curve $r = 1 - 2\sin\theta$.



(b) Find $\frac{dy}{dx}$ for this curve, in terms of θ .

$$x(\theta) = (1 - 2\sin\theta)\cos\theta = \cos\theta - 2\sin\theta\cos\theta$$

$$y(\theta) = (1 - 2\sin\theta)\sin\theta = \sin\theta - 2\sin^2\theta$$

$$x'(\theta) = -\sin\theta - 2(\cos^2\theta - \sin^2\theta)$$

$$y'(\theta) = \cos\theta - 4\sin\theta\cos\theta$$

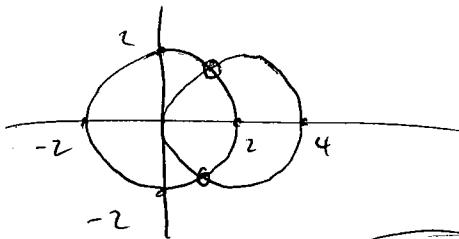
$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{\cos\theta - 4\sin\theta\cos\theta}{-\sin\theta - 2(\cos^2\theta - \sin^2\theta)}$$

3(a) Find all intersection points between the curves $r = 2$ and $r = 4 \cos \theta$.

$$\text{Set } 2 = 4 \cos \theta \Rightarrow \cos \theta = \frac{1}{2}, \theta = \frac{\pi}{3}, \frac{5\pi}{3}.$$

Check the origin - $r=2$ does not meet the origin.

Picture :



Points of intersection : $(2, \frac{\pi}{3})$ and $(2, \frac{5\pi}{3})$

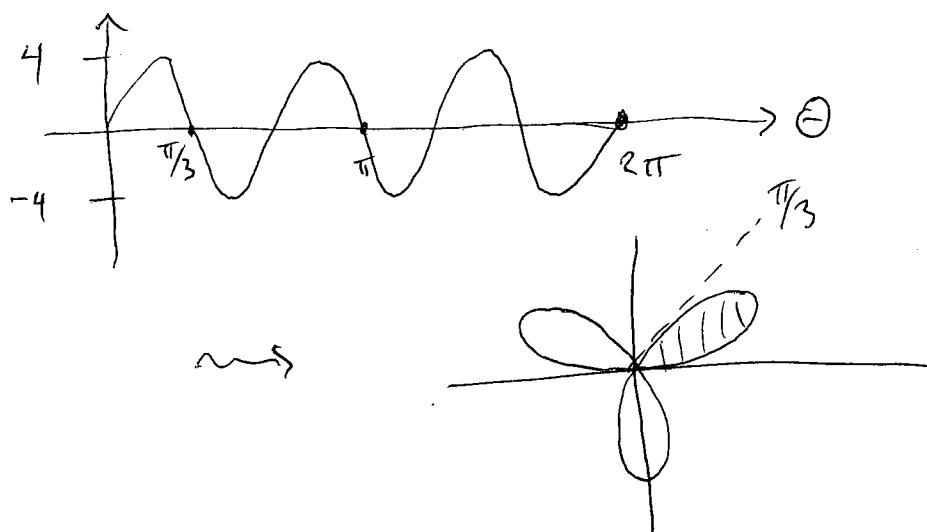
(b) Find a polar equation for the curve $xy = 4$.

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$r^2 \cos \theta \sin \theta = 4$$

4. Find the area enclosed by one loop of the curve $r = 4 \sin(3\theta)$. (Draw a picture!)



So one loop is
 $0 \leq \theta \leq \frac{\pi}{3}$

$$A = \int_0^{\pi/3} \frac{1}{2} r^2 d\theta = \int_0^{\pi/3} \frac{1}{2} (16 \sin^2(3\theta)) d\theta$$

$$= 8 \int_0^{\pi/3} \frac{1}{2} (1 - \cos(6\theta)) d\theta$$

$$= 4 \int_0^{\pi/3} (1 - \cos(6\theta)) d\theta$$

$$= 4 \left[\frac{\pi}{3} - \left[\frac{1}{6} \sin(6\theta) \right]_0^{\pi/3} \right]$$

$$= \frac{4\pi}{3} - \left(\frac{1}{6} \sin(2\pi) - \frac{1}{6} \sin(0) \right)$$

$$= \boxed{\frac{4\pi}{3}}$$

5. Find an equation of the tangent line to the curve $x = 10 - t^2$, $y = t^3 - 12t$ at the point $(6, -16)$.

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{3t^2 - 12}{-2t}$$

what is t at $(6, -16)$?

$$6 = 10 - t^2 \Rightarrow t^2 = 4 \Rightarrow t = \pm 2$$

check other coord: $-16 = (2)^3 - 12(2) ?$

yes

$$t=2 \checkmark$$

$$-16 = (-2)^3 - 12(-2) ?$$

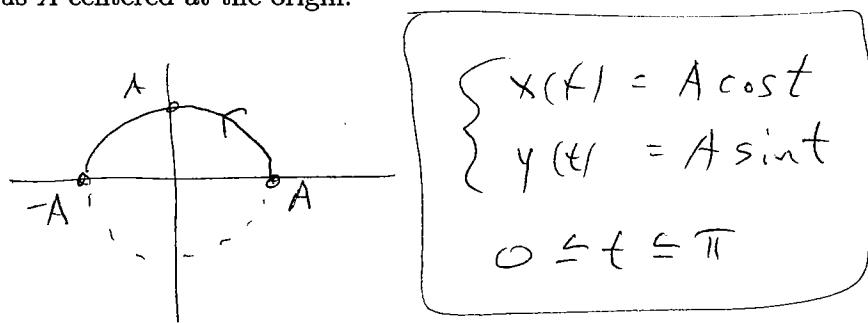
no

$$t \neq -2$$

At $t=2$, slope is $\frac{3(2)^2 - 12}{-2(2)} = 0$.

So line is $y = -16$ (slope zero, through $(6, -16)$)

- 6(a) Write down a parametrization (together with a "time" interval) for the upper half of a circle of radius A centered at the origin.



- (b) Rotate this curve about the x -axis and use an integral to find its surface area. What surface is this?

$$SA = \int_0^{\pi} 2\pi y \, ds$$

$$ds = \sqrt{x'(t)^2 + y'(t)^2} dt = \sqrt{A^2 \sin^2 t + A^2 \cos^2 t} dt = A dt$$

$$SA = \int_0^{\pi} 2\pi A \sin t \cdot A dt$$

$$= 2\pi A^2 \left[-\cos t \right]_0^{\pi}$$

$$= 2\pi A^2 [1 + 1]$$

$$= 4\pi A^2$$

, surface area of a sphere
of radius A .