3

1(a) Find the values of $x$ such that the vectors $\langle 4, x, 4\rangle$ and $\langle 2x, x, 5\rangle$ are orthogonal.

$$\langle 4, x, 3\rangle \cdot \langle 2x, x, 5\rangle = 8x + x^2 + 15 = 0$$

$$(x+3)(x+5) = 0$$

$$\boxed{x = -3 \quad \text{and} \quad x = -5}$$

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For $\langle 4, x, 4\rangle$ and $\langle 2x, x, 5\rangle$:

$$\langle 4, x, 4\rangle \cdot \langle 2x, x, 5\rangle = 8x + x^2 + 20 = 0$$

$$x = \frac{-8 \pm \sqrt{64 - 80}}{2} \quad \leftarrow \text{negative}$$

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so $\boxed{\text{there is no } x \text{ making these orthogonal}}$

(b) A parallelepiped has one corner at the origin and three adjacent corners at the points $(2, 1, 2)$, $(3, 3, 0)$ and $(1, 1, 1)$. Find the volume of the parallelepiped.

As vectors, the three sides originating from $(0,0,0)$ are $\underline{u} = \langle 2, 1, 2\rangle$, $\underline{v} = \langle 3, 3, 0\rangle$, $\underline{w} = \langle 1, 1, 1\rangle$.

Use $\text{vol} = \pm \underline{u} \cdot (\underline{v} \times \underline{w})$

$$\underline{v} \times \underline{w} = \langle 3, -3, 0\rangle$$

$$\underline{u} \cdot (\underline{v} \times \underline{w}) = 2 \cdot 3 + 1 \cdot (-3) + 2 \cdot 0$$

$$= \boxed{3}$$

**2(a)** Find a power series expansion, centered at 0, for the function $f(x) = \frac{1}{x+5}$.
**(b)** What is the radius of convergence of this power series?
**(c)** What is $f^{(50)}(0)$?

(a) $f(x) = \frac{1}{5+x} = \frac{1/5}{1/5(5+x)} = \frac{1/5}{1-\left(\frac{-x}{5}\right)}$

$= \sum_{n=1}^{\infty} (1/5)\left(\frac{-x}{5}\right)^{n-1}$ geometric series

$= \sum_{n=0}^{\infty} \left(\frac{1}{5}\right)\left(-\frac{x}{5}\right)^{n} = \boxed{\sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^{n+1} (-1)^n x^n}$.

(b) converges when $\left|\frac{-x}{5}\right| < 1$

i.e. $-5 < x < 5$

so radius $= \boxed{5}$

(c) the coefficient next to $x^n$ is

$\frac{f^{(n)}(0)}{n!} = \left(\frac{1}{5}\right)^{n+1} (-1)^n$.

So, $\boxed{f^{(50)}(0) = 50!\left(\frac{1}{5}\right)^{51}}$

**3(a)** Find two unit vectors that are orthogonal to both $3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{i} - 2\mathbf{k}$.

direction: use $\pm(\langle 3,-2,1\rangle \times \langle 1,0,-2\rangle)$

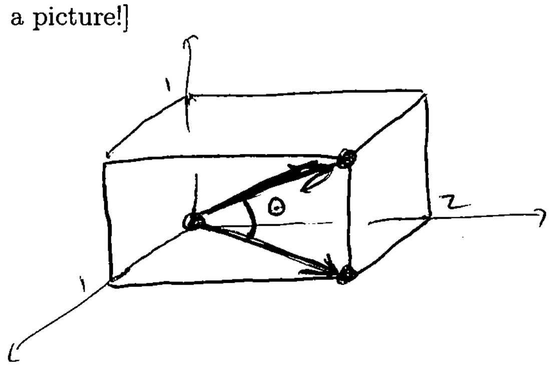
$= \pm\langle 4,7,2\rangle$

length $= \sqrt{16+49+4} = \sqrt{69}$

two unit vectors:

$$\boxed{\left\langle \frac{4}{\sqrt{69}}, \frac{7}{\sqrt{69}}, \frac{2}{\sqrt{69}} \right\rangle \text{ and } \left\langle \frac{-4}{\sqrt{69}}, \frac{-7}{\sqrt{69}}, \frac{-2}{\sqrt{69}} \right\rangle}$$

**(b)** A rectangular box has side lengths 1, 1, and 2. Find the cosine of the angle between the diagonal of the box and the diagonal of the long rectangular face (these diagonals meet at a corner). [Draw a picture!]




As vectors, diagonals are $\langle 1,2,1\rangle$ and $\langle 1,2,0\rangle$.

$$\cos\theta = \frac{\langle 1,2,1\rangle \cdot \langle 1,2,0\rangle}{|\langle 1,2,1\rangle|\,|\langle 1,2,0\rangle|} = \frac{5}{\sqrt{6}\,\sqrt{5}}$$

$$= \boxed{\frac{5}{\sqrt{30}}}$$

**4(a)** Write down the MacLaurin series for $e^x$.
**(b)** Write down $\frac{e^x}{x}$ as an infinite series.
**(c)** Write down $\int \frac{e^x}{x}\,dx$ as an infinite series.

(a) $$e^x = \boxed{\sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}$$

(b) $$\frac{e^x}{x} = \boxed{\sum_{n=0}^{\infty} \frac{1}{n!} \frac{x^n}{x} = \frac{1}{x} + 1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} + \cdots}$$

(c) $$\int \frac{e^x}{x}\,dx = C + \ln|x| + x + \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 3!} + \frac{x^4}{4 \cdot 4!} + \cdots$$

$$= C + \ln|x| + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$$

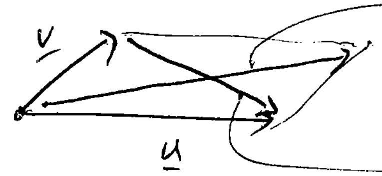
**5.** Let **u** and **v** be the sides of a parallelogram, considered as vectors, with initial point at the same corner.

**(a)** Express the diagonals of the parallelogram as vectors.

**(b)** Use properties of the dot product to express the squared-lengths of the diagonals in terms of $|\mathbf{u}|$, $|\mathbf{v}|$, and $\mathbf{u}\cdot\mathbf{v}$.

**(c)** Show that if the lengths of the diagonals are equal, then $\mathbf{u}\cdot\mathbf{v} = 0$. What does this say about the parallelogram?

(a)




$$\boxed{\begin{aligned} &\underline{u} + \underline{v} \\ &\underline{u} - \underline{v} \end{aligned}}$$

(b)

$$|\underline{u}+\underline{v}|^2 = (\underline{u}+\underline{v})\cdot(\underline{u}+\underline{v}) = (\underline{u}+\underline{v})\cdot\underline{u} + (\underline{u}+\underline{v})\cdot\underline{v}$$

$$= \underline{u}\cdot\underline{u} + \underline{v}\cdot\underline{u} + \underline{u}\cdot\underline{v} + \underline{v}\cdot\underline{v}$$

$$\boxed{|\underline{u}+\underline{v}|^2 = |\underline{u}|^2 + 2\,\underline{u}\cdot\underline{v} + |\underline{v}|^2}$$

$$|\underline{u}-\underline{v}|^2 = (\underline{u}-\underline{v})\cdot(\underline{u}-\underline{v}) = (\underline{u}-\underline{v})\cdot\underline{u} - (\underline{u}-\underline{v})\cdot\underline{v}$$

$$= \underline{u}\cdot\underline{u} - \underline{v}\cdot\underline{u} - \underline{u}\cdot\underline{v} + \underline{v}\cdot\underline{v}$$

$$\boxed{|\underline{u}-\underline{v}|^2 = |\underline{u}|^2 - 2\,\underline{u}\cdot\underline{v} + |\underline{v}|^2}$$

(c) If $|\underline{u}+\underline{v}| = |\underline{u}-\underline{v}|$

then $|\underline{u}+\underline{v}|^2 = |\underline{u}-\underline{v}|^2$

so $|\underline{u}|^2 + 2\,\underline{u}\cdot\underline{v} + |\underline{v}|^2 = |\underline{u}|^2 - 2\,\underline{u}\cdot\underline{v} + |\underline{v}|^2$

$$2\,\underline{u}\cdot\underline{v} = -2\,\underline{u}\cdot\underline{v}$$

$$4\,\underline{u}\cdot\underline{v} = 0$$

$$\underline{u}\cdot\underline{v} = 0$$

So — it is a rectangle.