

1. Test the following series for convergence or divergence. State the test you are using, and verify that the requirements are met. When possible, say whether it is conditionally convergent or absolutely convergent.

(a) $\sum_{n=1}^{\infty} \frac{4 + \cos n}{n^3}$ Comparison with $\sum_{n=1}^{\infty} \frac{5}{n^3}$:

$$\frac{4 + \cos n}{n^3} \leq \frac{4 + 1}{n^3} = \frac{5}{n^3}. \text{ Also, } \sum_{n=1}^{\infty} \frac{1}{n^3} \text{ converges}$$

(p-series) so $\sum_{n=1}^{\infty} \frac{5}{n^3} = 5 \cdot \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

Comp. Test $\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{4 + \cos n}{n^3} \text{ converges, absolutely}}$

(b/c terms are positive)

(b) $\sum_{n=1}^{\infty} \frac{1}{n^{n/2}}$

Root Test:

$$\sqrt[n]{|a_n|} = \left(\frac{1}{n^{n/2}}\right)^{1/n} = \frac{1}{n^{1/2}} \rightarrow 0 \text{ as } n \rightarrow \infty$$

Since $0 < 1$, Root test gives absolute convergence of the series.

(c) $\sum_{n=1}^{\infty} \frac{2n^2 + 1}{2n^{8/3} - 1}$

Compare with $\sum_{n=1}^{\infty} \frac{n^2}{n^{8/3}} = \sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$:

$$\frac{2n^2 + 1}{2n^{8/3} - 1} \geq \frac{2n^2}{2n^{8/3}} = \frac{n^2}{n^{8/3}} = \frac{1}{n^{2/3}},$$

and $\sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$ diverges (p-series, $p < 1$)

So

$$\boxed{\sum_{n=1}^{\infty} \frac{2n^2 + 1}{2n^{8/3} - 1} \text{ also diverges.}}$$

2(a) Find the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{n^2 + 4n + 4} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{(n+2)^2}$, telescoping series

$$= \left(\frac{1}{1^2} - \frac{1}{3^2} \right) + \left(\frac{1}{2^2} - \frac{1}{4^2} \right) + \left(\frac{1}{3^2} - \frac{1}{5^2} \right) + \dots$$

$$= 1 + \frac{1}{4} = \boxed{\frac{5}{4}}$$

all other terms cancel in pairs.

(b) The terms of a series are given by $a_1 = \frac{3}{5}$, $a_{n+1} = \frac{5n-3}{3n-5} a_n$. Determine whether the series converges or diverges.

Try Ratio Test!

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{5n-3}{3n-5} a_n \div a_n \right| = \left| \frac{5n-3}{3n-5} \right| = \left| \frac{5 - \frac{3}{n}}{3 - \frac{5}{n}} \right|$$

$$\rightarrow \frac{5}{3} \text{ as } n \rightarrow \infty.$$

Since $\frac{5}{3} > 1$, the Ratio Test says

the series diverges.

3(a) A ball is dropped from a height of 10 feet. Each time it strikes the ground it bounces vertically to a height that is $\frac{3}{4}$ of the preceding height. Use a geometric series to find the total distance the ball travels if it is allowed to bounce infinitely many times.

heights = $10, (\frac{3}{4}) \cdot 10, (\frac{3}{4})^2 \cdot 10, (\frac{3}{4})^3 \cdot 10, \text{ etc.}$

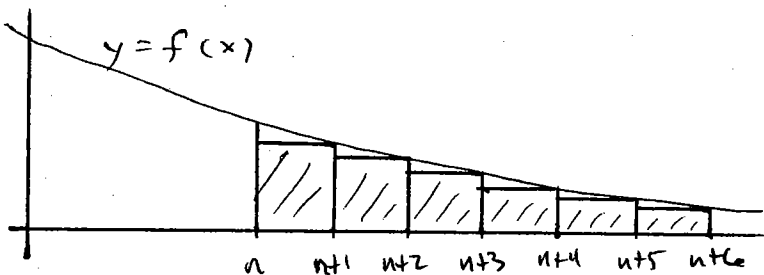
Total distance = twice sum of heights, minus 10
(first 10 ft. only happens once)

$$D = (20 + 20(\frac{3}{4}) + 20(\frac{3}{4})^2 + 20(\frac{3}{4})^3 + \dots) - 10$$

$$= (\text{geom. series with } a=20, r=\frac{3}{4}) - 10$$

$$= \frac{20}{1 - \frac{3}{4}} - 10 = 80 - 10 = \boxed{70 \text{ feet}}$$

(b) Recall that the n^{th} remainder of the series $\sum_{n=1}^{\infty} a_n$ is given by $R_n = a_{n+1} + a_{n+2} + \dots$. Suppose $f(x)$ is positive and decreasing for all x , and $a_n = f(n)$ for all n . The picture below illustrates an inequality between R_n and an improper integral. Give the inequality, and say briefly why it holds. [Hint: you will want to label some points on the picture correctly.]



Idea - areas of rectangles
are A_{n+1}, A_{n+2}, \dots

$R_n = \text{area of rectangles,}$

Area under curve = $\int_n^{\infty} f(x) dx$, plus area encloses
the rectangles.

So

$$R_n \leq \int_n^{\infty} f(x) dx$$

4. Find the radius of convergence and the interval of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{3^n (n+1)}$$

Try Ratio Test:

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(-1)^{n+1} x^{n+1}}{3^{n+1} (n+2)} \cdot \frac{3^n (n+1)}{(-1)^n x^n} \right| = \left| \frac{-x (n+1)}{3 (n+2)} \right|$$

$$= \left| \frac{x}{3} \right| \frac{n+1}{n+2} \rightarrow \left| \frac{x}{3} \right| \text{ as } n \rightarrow \infty$$

Test says:

Converge when $\left| \frac{x}{3} \right| < 1$, i.e. $-3 < x < 3$

Diverges when $\left| \frac{x}{3} \right| > 1$, i.e. $x > 3$ or $x < -3$

So, radius of convergence = 3.

At $x=3$ get $\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{3^n (n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$, converges

by Alt. series test.

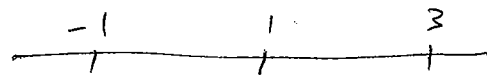
At $x=-3$ get $\sum_{n=0}^{\infty} \frac{(-1)^n (-3)^n}{3^n (n+1)} = \sum_{n=0}^{\infty} \frac{1}{n+1}$, diverges

(harmonic series).

So interval of convergence is $-3 < x \leq 3$.

5(a) Suppose the power series $\sum_{n=0}^{\infty} c_n(x-1)^n$ converges at $x=3$ and diverges at $x=-1$. What can you say about the interval of convergence?

I.o.C. is centered at 1.



Convergence at 3 \Rightarrow radius is ≥ 2

Divergence at -1 \rightarrow radius is ≤ 2

So radius of convergence = 2

Also, we know endpoint behavior already.

Interval is : $-1 < x \leq 3$

(b) Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$. First, write out several terms of the series. Then, how many terms must be added so that you can be sure the remainder is less than $\frac{1}{80}$? Explain how you know this (including any conditions that are required to hold).

Series is

$$\begin{aligned} & -\frac{1}{1} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{6^2} - \dots \\ & = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \frac{1}{36} - \dots \end{aligned}$$

By Alt. Series test (since $\frac{1}{n^2}$ is decreasing, with limit 0)

the remainder is \leq |next term|.

So the sum $-1 + \frac{1}{4} - \dots - \frac{1}{49} + \frac{1}{64}$ will have

remainder $\leq \frac{1}{81} < \frac{1}{80}$.

first 8 terms