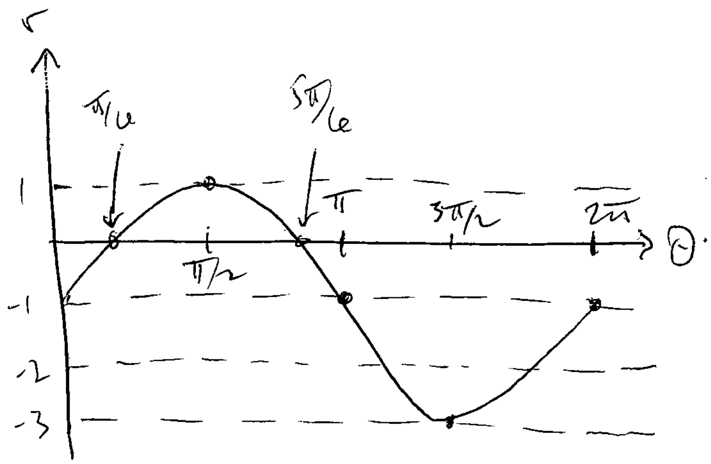
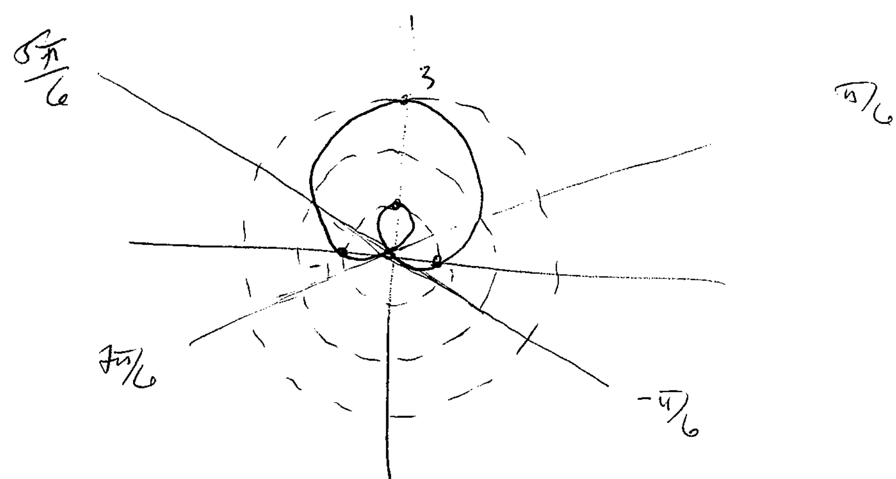
1(a) Graph carefully the polar curve $r = 2\sin\theta - 1$. [First draw the graph in the $r\theta$-plane. What are its intercepts, maxima and minima, etc?]

(b) Write down an expression (possibly involving definite integrals) for the area of the region between the two loops of your curve. [Do not evaluate this area.]





$$A = \int_{5\pi/6}^{13\pi/6} \tfrac{1}{2}(2\sin\theta - 1)^2\,d\theta \;-\; \int_{\pi/6}^{5\pi/6} \tfrac{1}{2}(2\sin\theta - 1)^2\,d\theta$$

2. For each of the series below, indicate whether it diverges, converges conditionally, or converges absolutely. Also say briefly how you know this.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$

a_n compare with $b_n = \frac{1}{\sqrt{n}}$

$\frac{a_n}{b_n} = \frac{\sqrt{n}}{\sqrt{n+1}} \to 1$ as $n \to \infty$

Limit comp. test $\Rightarrow \sum a_n, \sum b_n$ have same behavior (since $0 < 1 < \infty$).
Also $\sum b_n$ diverges (p-series, $p = \frac{1}{2}$)

Hence $\boxed{\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}} \text{ diverges}}$.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$ Alternating Series $\sum (-1)^n b_n$ with $b_n = \frac{1}{n^{3/2}}$. Note, the sequence b_n is decreasing, positive, and has limit 0.
Alt. Series Test $\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/2}}$ converges.

Also, $\left|\frac{(-1)^n}{n^{3/2}}\right| = \frac{1}{n^{3/2}}$ and $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges (p-series, $p = 3/2$).

So, the series $\boxed{\text{converges absolutely}}$.

3. Consider the two planes $-x-y-z=-1$ and $2x-4y+8z=10$.

(a) Find the direction of the line of intersection of the two planes.

(b) Find the cosine of the angle between the two planes.

The normal vectors to these planes are $n_1 = \langle -1,-1,-1 \rangle$ and $n_2 = \langle 2,-4,8 \rangle$.

The line is perpendicular to both, so its direction agrees with $n_1 \times n_2$.

$$n_1 \times n_2 = \boxed{\langle -12, 6, 6 \rangle}.$$

The angle θ between the planes is the same as the angle between n_1 and n_2.

So, $$\cos\theta = \frac{n_1 \cdot n_2}{|n_1||n_2|}$$

$$= \frac{-6}{\sqrt{3}\sqrt{84}} = \boxed{\frac{-6}{\sqrt{252}}}$$

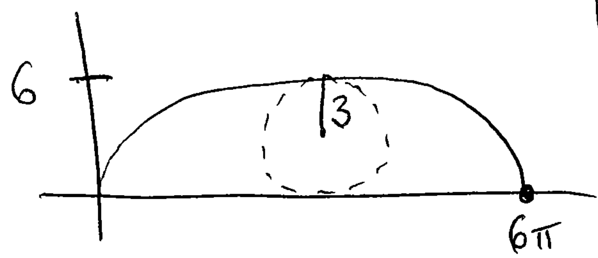
4. Recall that the *cycloid* generated by a circle of radius 3 has parametric equations

$$x(t) = 3(t - \sin t), \quad y(t) = 3(1 - \cos t).$$

(a) Draw carefully one arch of the cycloid, indicating its height, its endpoints, and the t interval it corresponds to.

(b) Write down a definite integral giving the length of one arch of this cycloid.

(c) Calculate this length. [Hint: a trig identity will help.]



$$\boxed{\begin{aligned} \text{height} &= \text{diameter of circle} \\ &= 6 \\ 0 \le t &\le 2\pi \end{aligned}}$$

$x'(t) = 3(1-\cos t)$, $\quad y'(t) = 3\sin t$

$$ds = \sqrt{9(1-\cos t)^2 + 9\sin^2 t}\,dt$$

$$= 3\sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\,dt$$

$$= 3\sqrt{2 - 2\cos t}\,dt = 3\sqrt{2}\sqrt{1-\cos t}\,dt$$

So $L = \boxed{\int_0^{2\pi} 3\sqrt{2}\sqrt{1-\cos t}\,dt}$

evaluating: use $1 - \cos t = 2\sin^2 \frac{t}{2}$

$$L = \int_0^{2\pi} 3\sqrt{2}\sqrt{2\sin^2 \tfrac{t}{2}}\,dt = \int_0^{2\pi} 6 \sin \tfrac{t}{2}\,dt$$

$$= -12 \cos \tfrac{t}{2} \Big|_0^{2\pi}$$

$$\boxed{L = 24}$$

5. Find $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ for the curve $\mathbf{r}(t) = \langle 4\sin t, 3t, 4\cos t\rangle$.

$$r'(t) = \langle 4\cos t, 3, -4\sin t\rangle$$

$$|r'(t)| = \sqrt{16\cos^2 t + 9 + 16\sin^2 t} = \sqrt{25} = 5$$

$$\boxed{T(t) = \frac{r'(t)}{|r'(t)|} = \left\langle \frac{4}{5}\cos t, \frac{3}{5}, \frac{-4}{5}\sin t \right\rangle}$$

$$T'(t) = \left\langle \frac{-4}{5}\sin t, 0, \frac{-4}{5}\cos t \right\rangle$$

$$|T'(t)| = \sqrt{\frac{16}{25}\sin^2 t + \frac{16}{25}\cos^2 t} = \frac{4}{5}$$

$$\boxed{N(t) = \frac{T'(t)}{|T'(t)|} = \langle -\sin t, 0, -\cos t\rangle}$$

$$B(t) = T(t) \times N(t)$$

$$\boxed{B(t) = \left\langle \frac{-3}{5}\cos t, \frac{4}{5}, \frac{3}{5}\sin t \right\rangle}$$

6(a) Write down the MacLaurin series for $\cos x$. Where does it converge?
(b) Write $\cos(x^4)$ as an infinite series. Where does it converge?
(c) Write $\int \cos(x^4)\,dx$ as an infinite series. Where does it converge?

$$\boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots}$$

converges for all x

$$\cos(x^4) = 1 - \frac{(x^4)^2}{2!} + \frac{(x^4)^4}{4!} - \frac{(x^4)^6}{6!} + \frac{(x^4)^8}{8!} - \frac{(x^4)^{10}}{10!} + \cdots$$

$$\boxed{\cos(x^4) = 1 - \frac{x^8}{2!} + \frac{x^{16}}{4!} - \frac{x^{24}}{6!} + \frac{x^{32}}{8!} - \frac{x^{40}}{10!} + \cdots}$$

converges for all x

$$\int \cos(x^4)\,dx =$$

$$C + x - \frac{x^9}{9\cdot 2!} + \frac{x^{17}}{17\cdot 4!} - \frac{x^{25}}{25\cdot 6!} + \frac{x^{33}}{33\cdot 8!} - \frac{x^{41}}{41\cdot 10!} + \cdots$$

converges for all x

OR:

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \qquad \cos(x^4) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n}}{(2n)!},$$

$$\int \cos(x^4)\,dx = \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+1}}{(8n+1)(2n)!} \right) + C$$

7(a) Calculate $|\mathbf{u} \times \mathbf{v}|$ if $|\mathbf{u}| = 2$, $\mathbf{u} \cdot \mathbf{v} = 3$, and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\pi/3$.

(b) Write down parametric equations for the line through the point $(4, 0, -4)$ and perpendicular to the plane $5x + 4 = 3y + 2z$.

(a) We know $u \cdot v = |u||v| \cos\theta$ ①

and $|u \times v| = |u||v| \sin\theta$ ②.

Since $\theta = \pi/3$, we get

① $3 = 2|v| \frac{1}{2}$

so $|v| = 3$.

Now, by ②, $|u \times v| = 2 \cdot 3 \cdot \frac{\sqrt{3}}{2} = \boxed{3\sqrt{3}}$

(b)

$\underline{r}(t) = \langle 4, 0, -4 \rangle + t \langle 5, -3, -2 \rangle$

↑ normal vector to the plane

So

$$\boxed{r(t) = \langle 4+5t, -3t, -4-2t \rangle}$$

8(a) Write down a parametrization of the circle of radius a centered at the origin in the xy-plane.
(b) Find the curvature $\kappa(t)$ of this circle.

(a) $\boxed{r(t) = \langle a\cos t,\ a\sin t,\ 0\rangle, \quad 0 \le t \le 2\pi}$

(b) $r'(t) = \langle -a\sin t,\ a\cos t,\ 0\rangle$

$$|r'(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = a$$

$$r''(t) = \langle -a\cos t,\ -a\sin t,\ 0\rangle$$

$$r'(t) \times r''(t) = \langle 0,\ 0,\ \underbrace{a^2\sin^2 t + a^2\cos^2 t}_{=a^2} \rangle$$

$$|r'(t) \times r''(t)| = a^2$$

so

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{a^2}{a^3} = \boxed{\frac{1}{a}}$$