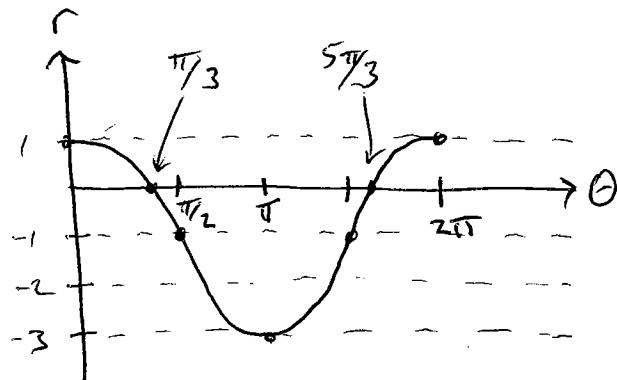


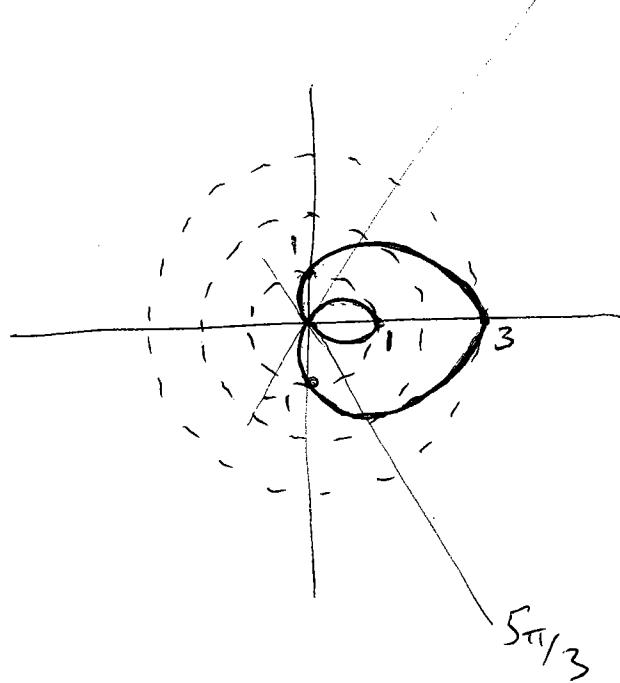
1(a) Graph carefully the polar curve $r = 2 \cos \theta - 1$. [First draw the graph in the $r\theta$ -plane. What are its intercepts, maxima and minima, etc?]

(b) Write down an expression (possibly involving definite integrals) for the area of the region between the two loops of your curve. [Do not evaluate this area.]



$$r=0 \text{ when } \cos \theta = \frac{1}{2}$$

$$\text{i.e. } \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$



Region :



$$\text{Area} = \int_{\pi/3}^{5\pi/3} \frac{1}{2} |r|^2 d\theta - \int_{-\pi/3}^{\pi/3} \frac{1}{2} |r|^2 d\theta$$

$$\text{Area} = \int_{\pi/3}^{5\pi/3} \frac{1}{2} (2 \cos \theta - 1)^2 d\theta - \int_{-\pi/3}^{\pi/3} \frac{1}{2} (2 \cos \theta - 1)^2 d\theta$$

2. For each of the series below, indicate whether it diverges, converges conditionally, or converges absolutely. Also say briefly how you know this.

$$(a) \sum_{n=2}^{\infty} \frac{1}{(n-1)^2}$$

a_n , compare with $b_n = \frac{1}{n^2}$

$$\frac{a_n}{b_n} = \frac{n^2}{(n-1)^2} = \frac{n^2}{n^2 - 2n + 1} \rightarrow 1 \text{ as } n \rightarrow \infty.$$

Since $0 < 1 < \infty$, Limit Comparison test says $\sum a_n, \sum b_n$ have same behavior. Since $\sum \frac{1}{n^2}$ converges (p-series), so does

$\sum \frac{1}{(n-1)^2}$. Terms are positive, so it

[converges absolutely].

$$(b) \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

note that $\left| \frac{(-1)^n}{\sqrt{n}} \right| = \frac{1}{\sqrt{n}}$ which is decreasing, with limit 0 as $n \rightarrow \infty$.

Alternating Series Test \Rightarrow series converges.

But, $\sum \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum \frac{1}{\sqrt{n}}$ diverges (p-series, $p \leq 1$)

So, the series [converges conditionally].

3. Consider the two planes $x + y + z = 1$ and $4x - 3y + 2z = 0$.

(a) Find the direction of the line of intersection of the two planes.

(b) Find the cosine of the angle between the two planes.

The normal vectors to these planes are

$$\mathbf{n}_1 = \langle 1, 1, 1 \rangle \text{ and } \mathbf{n}_2 = \langle 4, -3, 2 \rangle .$$

The line is perpendicular to both, so its direction agrees with $\mathbf{n}_1 \times \mathbf{n}_2$.

$$\mathbf{n}_1 \times \mathbf{n}_2 = \boxed{\langle 5, 2, -7 \rangle} .$$

The angle θ between the planes is the same as the angle between \mathbf{n}_1 and \mathbf{n}_2 .

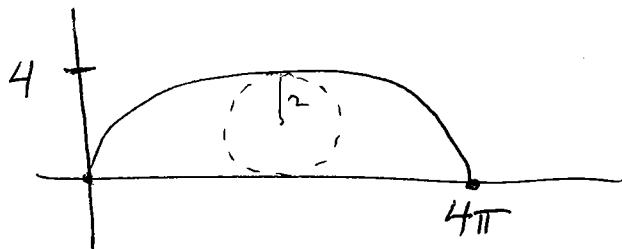
$$\text{So, } \cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|}$$

$$= \frac{3}{\sqrt{3} \sqrt{16+9+4}} = \boxed{\frac{3}{\sqrt{87}}}$$

4. Recall that the *cycloid* generated by a circle of radius 2 has parametric equations

$$x(t) = 2(t - \sin t), \quad y(t) = 2(1 - \cos t).$$

- (a) Draw carefully one arch of the cycloid, indicating its height, its endpoints, and the t interval it corresponds to.
 (b) Write down a definite integral giving the length of one arch of this cycloid.
 (c) Calculate this length. [Hint: a trig identity will help.]



height = diameter of circle
 = 4
 $0 \leq t \leq 2\pi$

$$x'(t) = 2(1 - \cos t), \quad y'(t) = 2\sin t$$

$$\sqrt{s} = \sqrt{4(1 - \cos t)^2 + 4\sin^2 t} dt$$

$$= 2 \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} dt$$

$$= 2 \sqrt{2 - 2\cos t} dt = 2\sqrt{2} \sqrt{1 - \cos t} dt$$

So $L = \int_0^{2\pi} 2\sqrt{2} \sqrt{1 - \cos t} dt$

evaluating: $1 - \cos t = 2 \sin^2 \frac{t}{2}$

$$L = \int_0^{2\pi} 2\sqrt{2} \sqrt{2 \sin^2 \frac{t}{2}} dt = \int_0^{2\pi} 4 \sin \frac{t}{2} dt = -8 \cos \frac{t}{2} \Big|_0^{2\pi}$$

$L = 16$

5. Find $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ for the curve $\mathbf{r}(t) = \langle 3 \sin t, 4t, 3 \cos t \rangle$.

$$\mathbf{r}'(t) = \langle 3 \cos t, 4, -3 \sin t \rangle$$

$$|\mathbf{r}'(t)| = \sqrt{9 \cos^2 t + 16 + 9 \sin^2 t} = 5$$

$$\boxed{\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \left\langle \frac{3}{5} \cos t, \frac{4}{5}, \frac{-3}{5} \sin t \right\rangle}$$

$$\mathbf{T}'(t) = \left\langle -\frac{3}{5} \sin t, 0, -\frac{3}{5} \cos t \right\rangle$$

$$|\mathbf{T}'(t)| = \sqrt{\frac{9}{25} \sin^2 t + \frac{9}{25} \cos^2 t} = \frac{3}{5}$$

$$\boxed{\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \langle -\sin t, 0, -\cos t \rangle}$$

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\boxed{\mathbf{B}(t) = \left\langle -\frac{4}{5} \cos t, \frac{3}{5}, \frac{4}{5} \sin t \right\rangle}$$

6(a) Write down the MacLaurin series for $\sin x$. Where does it converge?

(b) Write $\sin(x^3)$ as an infinite series. Where does it converge?

(c) Write $\int \sin(x^3) dx$ as an infinite series. Where does it converge?

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \dots$$

converges for all x

$$\sin x^3 = x^3 - \frac{(x^3)^3}{3!} + \frac{(x^3)^5}{5!} - \frac{(x^3)^7}{7!} + \frac{(x^3)^9}{9!} - \frac{(x^3)^{11}}{11!} + \dots$$

$$\sin x^3 = x^3 - \frac{x^9}{3!} + \frac{x^{15}}{5!} - \frac{x^{21}}{7!} + \frac{x^{27}}{9!} - \frac{x^{33}}{11!} + \dots$$

converges for all x

$$\int \sin(x^3) dx =$$

$$C + \frac{x^4}{4} - \frac{x^{10}}{10 \cdot 3!} + \frac{x^{16}}{16 \cdot 5!} - \frac{x^{22}}{22 \cdot 7!} + \frac{x^{28}}{28 \cdot 9!} - \frac{x^{34}}{34 \cdot 11!} + \dots$$

converges for all x

OR: $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$ $\sin x^3 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+3}}{(6n+3)!}$

$$\int \sin(x^3) dx = \left(\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+4}}{(6n+4) \cdot (2n+1)!} \right) + C.$$

7(a) Calculate $|\mathbf{u} \times \mathbf{v}|$ if $|\mathbf{u}| = 2$, $\mathbf{u} \cdot \mathbf{v} = 3$, and the angle between \mathbf{u} and \mathbf{v} is $\pi/6$.

(b) Write down parametric equations for the line through the point $(2, 3, 4)$ and perpendicular to the plane $2x + 3y = 4z + 5$.

(a) We know $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta$ ①
 and $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}| |\mathbf{v}| \sin \theta$ ②

Since $\theta = \pi/6$, we get

$$\textcircled{1} \quad 3 = 2 |\mathbf{v}| \frac{\sqrt{3}}{2}$$

$$\text{so } |\mathbf{v}| = \frac{3}{\sqrt{3}}$$

Now, by ②, $|\mathbf{u} \times \mathbf{v}| = 2 \cdot \frac{3}{\sqrt{3}} \cdot \frac{1}{2} = \frac{3}{\sqrt{3}}$

(b)

$$\underline{s}(t) = \langle 2, 3, 4 \rangle + t \langle 2, 3, -4 \rangle$$

\nwarrow normal vector to
plane

so $\underline{s}(t) = \langle 2+2t, 3+3t, 4-4t \rangle$

- 8(a) Write down a parametrization of the circle of radius a centered at the origin in the xy -plane.
 (b) Find the curvature $\kappa(t)$ of this circle.

(a) $\underline{r}(t) = \langle a \cos t, a \sin t, 0 \rangle, 0 \leq t \leq 2\pi$

(b) $r'(t) = \langle -a \sin t, a \cos t, 0 \rangle$

$$|r'(t)| = \sqrt{a^2 \sin^2 t + a^2 \cos^2 t} = a$$

$$r''(t) = \langle -a \cos t, -a \sin t, 0 \rangle$$

$$r'(t) \times r''(t) = \langle 0, 0, \underbrace{a^2 \sin^2 t + a^2 \cos^2 t}_{= a^2} \rangle$$

$$|r'(t) \times r''(t)| = a^2$$

So

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{a^2}{a^3} = \boxed{\frac{1}{a}}.$$