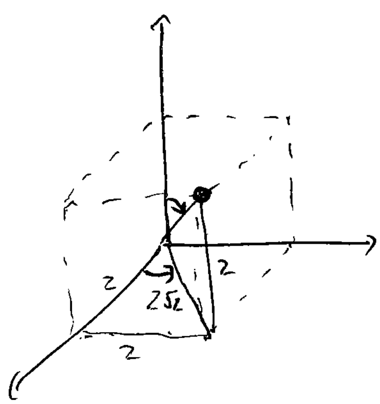
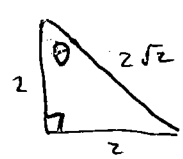
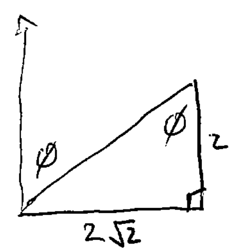
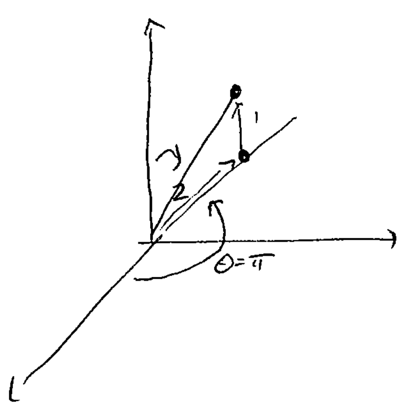
1. (10 points) Each row in the table below represents a point in three dimensions. Express each point in rectangular, cyclindrical, and spherical coordinates, and fill in the empty boxes accordingly. [It may help to draw pictures.]

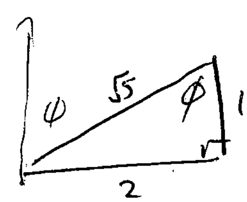
rectangular (x, y, z)	cylindrical (r, θ, z)	spherical (ρ, θ, ϕ)
$(2, 2, 2)$	$(2\sqrt{2}, \frac{\pi}{4}, 2)$	$(2\sqrt{3}, \frac{\pi}{4}, \tan^{-1}\sqrt{2})$
$(-2, 0, 1)$	$(2, \pi, 1)$	$(\sqrt{5}, \pi, \tan^{-1}(2))$
$(0, 0, -5)$	$(0, \pi/7, -5)$	$(5, \pi/7, \pi)$

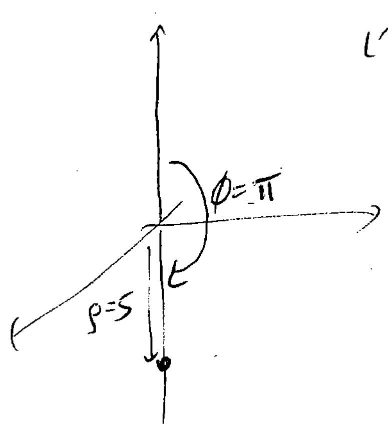








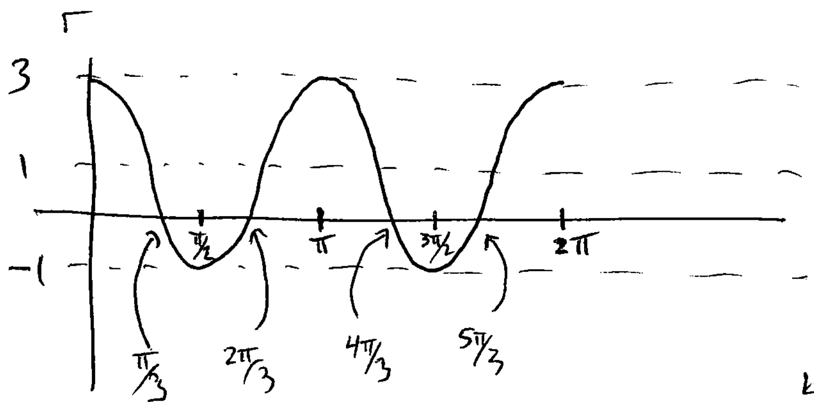




2. (10 points) Consider the polar curve $r = 1 + 2\cos 2\theta$.

(a) Carefully graph this curve, indicating any special angles, intercepts, etc. [First draw the graph in the $r\theta$-plane. Where exactly does it cross the θ-axis? What are its maxima and minima?]

(b) Write down an integral representing the area of one of the larger lobes of the curve.



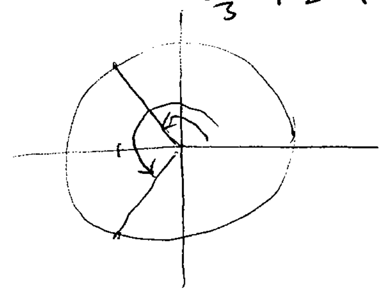
$r = 0$ when

$1 + 2\cos 2\theta = 0$

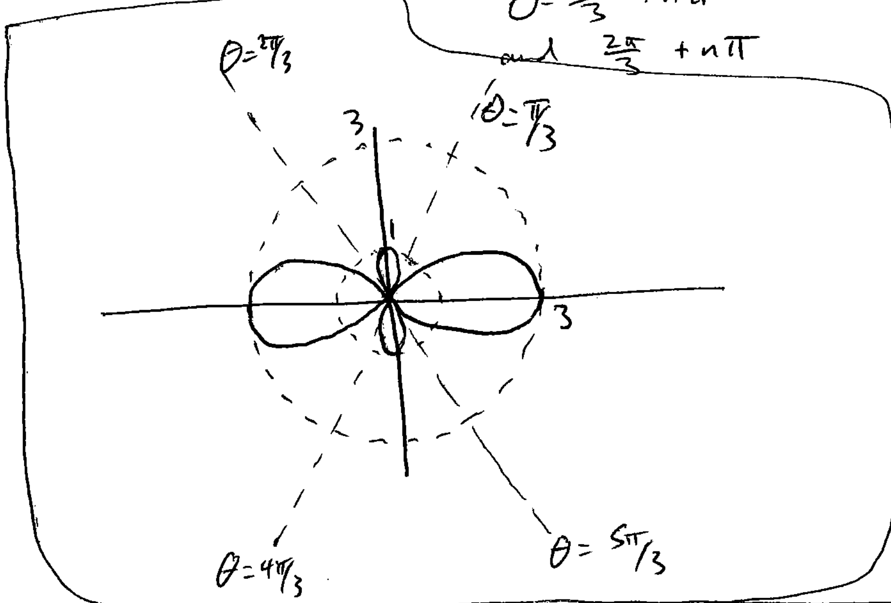
$2\cos 2\theta = -1$

$\cos 2\theta = \frac{-1}{2}$

$2\theta = \frac{2\pi}{3} + 2n\pi$ or $\frac{4\pi}{3} + 2n\pi$



$\theta = \frac{\pi}{3} + n\pi$ and $\frac{2\pi}{3} + n\pi$



Area of one large lobe $= \boxed{\displaystyle\int_{-\pi/3}^{\pi/3} \frac{1}{2}(1+2\cos 2\theta)^2 \, d\theta}$

3. (10 points) Consider the points $P = (1, 0, 0)$, $Q = (0, 2, 0)$, and $R = (0, 0, 3)$.

(a) Find an equation of the plane through P, Q, and R.

(b) Find the area of the triangle with corners P, Q, and R.

$\vec{PQ} = \langle -1, 2, 0 \rangle \qquad \vec{PR} = \langle -1, 0, 3 \rangle$

$PQ \times PR = \langle 6, 3, 2 \rangle$

$6x + 3y + 2z = C$

$6(1) + 3(0) + 2(0) = C \Rightarrow C = 6$

$\boxed{6x + 3y + 2z = 6}$

area = $\frac{1}{2}$ area of parallelogram

$= \frac{1}{2} |PQ \times PR|$

$= \frac{1}{2} \sqrt{36 + 9 + 4}$

$= \boxed{\frac{7}{2}}$

4. (10 points) Let $\mathbf{r}(t) = \langle t^2, \sin t - t\cos t, \cos t + t\sin t\rangle$ for $t > 0$.

(a) Find the unit tangent vector $\mathbf{T}(t)$ for $t > 0$.

(b) Find the curvature $\kappa(t)$ for $t > 0$.

$$\underline{r}'(t) = \langle 2t,\ \cos t + t\sin t - \cos t,\ -\sin t + t\cos t + \sin t\rangle$$

$$= \langle 2t,\ t\sin t,\ t\cos t\rangle$$

$$|\underline{r}'(t)| = \sqrt{4t^2 + t^2\sin^2 t + t^2\cos^2 t}$$

$$= \sqrt{5t^2} = \sqrt{5}\,t$$

$$\underline{T}(t) = \frac{\underline{r}'(t)}{|\underline{r}'(t)|} = \boxed{\left\langle \frac{2}{\sqrt{5}},\ \frac{\sin t}{\sqrt{5}},\ \frac{\cos t}{\sqrt{5}} \right\rangle}.$$

$$\underline{T}'(t) = \left\langle 0,\ \frac{\cos t}{\sqrt{5}},\ \frac{-\sin t}{\sqrt{5}} \right\rangle$$

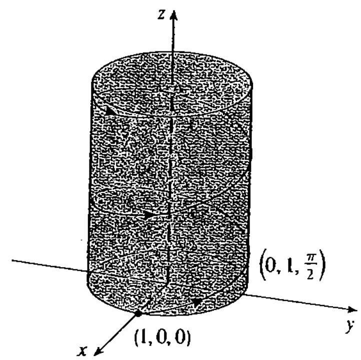
$$|\underline{T}'(t)| = \sqrt{\frac{\cos^2 t}{5} + \frac{\sin^2 t}{5}} = \frac{1}{\sqrt{5}}$$

$$\kappa(t) = \frac{|\underline{T}'(t)|}{|\underline{r}'(t)|} = \frac{1}{\sqrt{5}\cdot\sqrt{5}t} = \boxed{\frac{1}{5t}}$$

5. (10 points) The picture shows part of a helix on the cylinder of radius one, starting at $(1, 0, 0)$ and ending at $(1, 0, 4\pi)$.

(a) Write down a vector-valued function $\mathbf{r}(t)$ which traverses this curve.

(b) Using your expression for $\mathbf{r}(t)$, find the length of this curve.



$$\boxed{\begin{gathered}\underline{r}(t) = \langle \cos t, \sin t, t \rangle \\ 0 \le t \le 4\pi\end{gathered}}$$

$$\text{Length} = \int_0^{4\pi} |\underline{r}'(t)|\,dt$$

$$= \int_0^{4\pi} |\langle -\sin t, \cos t, 1 \rangle|\,dt$$

$$= \int_0^{4\pi} \sqrt{\sin^2 t + \cos^2 t + 1}\,dt$$

$$= \int_0^{4\pi} \sqrt{2}\,dt = \boxed{4\pi\sqrt{2}}$$

6(a) (5 points) Use the integral test to investigate convergence of the series $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$. Be sure to check that all the requirements of the test are satisfied. [Hint: substitution.]

(b) (5 points) Does the series $\sum_{n=2}^{\infty}(-1)^n \frac{n^2+1}{n^3-1}$ converge absolutely, converge conditionally, or diverge? State any tests you use and explain why they apply.

(a) $f(x) = \frac{1}{x\ln(x)}$ is continuous and positive.

$f'(x) = \frac{-1-\ln x}{(x\ln(x))^2}$, negative when $\ln x > 1$, i.e. when $x > e$, so $f(x)$ is eventually decreasing.

So series converges $\Leftrightarrow$ $\int_2^{\infty} \frac{1}{x\ln(x)}\,dx$ converges.

$$\int_2^{\infty} \frac{1}{x\ln x}\,dx = \int_{\ln 2}^{\infty} \frac{1}{u}\,du \qquad \begin{aligned} u &= \ln x \\ du &= \tfrac{1}{x}\,dx \end{aligned}$$

$$= \lim_{b\to\infty} \Big[\ln u\Big]_{\ln 2}^{b} = \lim_{b\to\infty}\left(\ln(b) - \ln(\ln(2))\right)$$

$= \infty$ so $\boxed{\text{diverges}}$

(b) Series converges by alt. series test,

Since $\frac{n^2+1}{n^3-1} > 0$, and $\frac{(n+1)^2+1}{(n+1)^3-1} \leq \frac{n^2+1}{n^3-1}$ for all n,

and $\lim_{n\to\infty} \frac{n^2+1}{n^3-1} = 0$. (criss-multiply to check)

Next consider $\sum_{n=2}^{\infty} \frac{n^2+1}{n^3-1}$. Note $\frac{n^2+1}{n^3-1} > \frac{1}{n}$ for all n

(since $n(n^2+1) > n^3-1$) so the series diverges by comparison with $\sum_{n=2}^{\infty}\frac{1}{n}$.

$\Rightarrow$ $\boxed{\text{Conditional convergence}}$.

7. (10 points) Find the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{5^n n^5}$. Be sure to check the endpoints.

Ratio Test: $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+1}}{5^{n+1}(n+1)^5} \cdot \frac{5^n n^5}{x^n}\right|$

$$= \left|\frac{x}{5} \cdot \underbrace{\left(\frac{n}{n+1}\right)^5}_{\to 1}\right|.$$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x}{5}\right|$ since $\left(\frac{n}{n+1}\right)^5 \to (1)^5 = 1$

So ratio test gives convergence when $\left|\frac{x}{5}\right| < 1$ and divergence when $\left|\frac{x}{5}\right| > 1$.

($\left|\frac{x}{5}\right| < 1 \iff -5 < x < 5$)

endpoints: $x = 5$: $\sum_{n=1}^{\infty} \frac{5^n}{5^n n^5} = \sum_{n=1}^{\infty} \frac{1}{n^5}$

converges by p-series test.

$x = -5$: $\sum_{n=1}^{\infty} (-1)^n \frac{5^n}{5^n n^5} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^5}$ converges

because its absolute value converges.

Interval of convergence:

$$\boxed{-5 \le x \le 5}$$

8. (5 points) Find the Taylor series for $f(x) = \cos x$ centered at $\pi/2$.

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f(\pi/2) = 0$

$f'(\pi/2) = -1$

$f''(\pi/2) = 0$ then these repeat

$f'''(\pi/2) = 1$

So Taylor series is

$$\frac{-1}{1!}(x - \pi/2) + \frac{3}{3!}(x-\pi/2)^3 + \frac{-5}{5!}(x-\pi/2)^5 + \frac{7}{7!}(x-\pi/2)^7 + \cdots$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-\pi/2)^{2n+1}}{(2n+1)!}}$$