2. Consider the power series $\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}}(x+3)^n$.

(a) When $x = -4$ does this series converge or diverge?

(b) Determine all the values of x for which the series converges.

(c) What is the domain of the function $f(x) = \sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}}(x+3)^n$?

(a) $\sum_{n=1}^{\infty} \frac{(-2)^n}{\sqrt{n}}(-1)^n = \sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}}$ terms $\to \infty$ so diverges by divergence test.

(or $\frac{2^n}{\sqrt{n}} \geq \frac{1}{\sqrt{n}}$ and $\sum \frac{1}{\sqrt{n}}$ diverges)

(or apply part (b))

(b) ratio test:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-2)^{n+1}(x+3)^{n+1}\sqrt{n}}{\sqrt{n+1}\,(-2)^n (x+3)^n}\right| = \left|\sqrt{\frac{n}{n+1}}\cdot 2(x+3)\right|$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\sqrt{\frac{n}{n+1}}\cdot 2|x+3| = 2|x+3|$$

$\Rightarrow$ Converges if $2|x+3| < 1$, or $|x+3| < \frac{1}{2}$,

or $-3-\frac{1}{2} < x < -3+\frac{1}{2}$

Endpoints

$x = -3-\frac{1}{2}$: $\sum_{n=0}^{\infty} \frac{(-2)^n}{\sqrt{n}}\left(\frac{-1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}$ diverges (p-series)

$x = -3+\frac{1}{2}$: $\sum_{n=0}^{\infty} \frac{(-2)^n}{\sqrt{n}}\left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges (alt. series test)

So converges when $\boxed{-3-\frac{1}{2} < x \leq -3+\frac{1}{2}}$.

(c) Domain $= \left(-\frac{7}{2}, -\frac{5}{2}\right]$.

3. Test the following series for convergence or divergence. State the test you are using, and verify that the requirements are met. If possible, say whether it is absolutely convergent or conditionally convergent.

(a) $\sum_{n=3}^{\infty} \frac{1}{n(\ln n)^{1000}}$

Integral Test: $\int_3^{\infty} \frac{1}{x(\ln x)^{1000}}\,dx$ $\quad u = \ln x$, $du = \frac{1}{x}dx$

$$= \int_{\ln 3}^{\infty} u^{-1000}\,du = \left.\frac{u^{-999}}{-999}\right|_{\ln 3}^{\infty}$$

$$= \lim_{b\to\infty}\left[\frac{-1}{999\ b^{999}} + \frac{1}{999(\ln 3)^{999}}\right] = \frac{1}{999(\ln 3)^{999}}.$$

Integral converges, so series converges.
Terms are positive, so absolute convergence.

(b) $\sum_{n=0}^{\infty} \frac{n^{2n}}{(1+2n^2)^n}$

root test: $\sqrt[n]{|a_n|} = \frac{n^2}{1+2n^2}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n^2}{1+2n^2} = \frac{1}{2} < 1$$

so converges absolutely.

(c) $\sum_{n=1}^{\infty} \frac{n^3 - \sqrt{n} + 6}{n^4 - n - 3}$

limit comparison with $\frac{1}{n}$
(direct comparison doesn't work)

so: $$\frac{n^3 - \sqrt{n} + 6}{n^4 - n - 3} \cdot \frac{n}{1} = \frac{n^4 - n^{3/2} + 6n}{n^4 - n - 3}$$

$$= \frac{1 - n^{-5/2} + \frac{6}{n^3}}{1 - \frac{1}{n^3} - \frac{3}{n^4}} \to 1$$ not 0 or ∞

$\Rightarrow$ same behavior as $\sum \frac{1}{n}$ which diverges.

$\Rightarrow$ series diverges.

4. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{16}{1000^n}$.

(a) Write the fifth partial sum of this series in decimal form.

(b) Determine the sum of the series as a quotient of two integers.

(a) $\frac{16}{1000} + \frac{16}{1000^2} + \frac{16}{1000^3} + \frac{16}{1000^4} + \frac{16}{1000^5}$

$= \boxed{0.016016016016016}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{16}{1000^n} = \frac{16}{1000} \sum_{n=1}^{\infty} \left(\frac{1}{1000}\right)^{n-1}$

geometric series with $a = \frac{16}{1000}$, $r = \frac{1}{1000}$.

($|r| < 1$)

So sum is $\frac{a}{1-r} =$

$\frac{16}{1000} \cdot \frac{1}{1 - \frac{1}{1000}} =$

$\frac{16}{1000} \cdot \frac{1000}{999} = \boxed{\frac{16}{999}}$

5. Consider the integral $\int x\cos(x^3)\,dx$.

(a) Write down the Maclaurin series for $\cos(x)$, $\cos(x^3)$, and $x\cos(x^3)$. Either use Σ notation or write down enough terms so that the pattern is perfectly clear.

(b) Evaluate the integral $\int x\cos(x^3)\,dx$ as an infinite series.

(a) $\cos(x) = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}}$

$$\cos(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n}}{(2n)!}$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!}}$$

$$x\cos(x^3) = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{(2n)!}}$$

(b) $\int x\cos(x^3)\,dx =$

$$\boxed{C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!}}$$

1. The real number π can be expressed as an infinite sum: $\pi = \frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} - \cdots$

(a) Express this formula for π using Σ notation.

(b) Find the sum of the series $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$

(c) Find the sum of the series $\frac{1}{7} - \frac{1}{9} + \frac{1}{11} - \frac{1}{13} + \cdots$

(d) Find the sum of the series $\sum_{n=-3}^{\infty} (-1)^{n+1} \frac{1}{2n+5}$

(a) $\pi = \sum_{n=0}^{\infty} (-1)^n \frac{4}{2n+1}$

(b) $\frac{\pi}{4}$ (same as ~~the~~ $\frac{\text{first series}}{4}$)

(c) Same as (b), but 3 terms missing and opposite signs.

So, by (b),

$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - (\text{sum of this series})$

$\Rightarrow \boxed{\text{sum} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{\pi}{4}}$

(d) Series is

$-1 - 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \frac{1}{11} - \cdots$

$= -1 - (\text{series for } \frac{\pi}{4})$

$= \boxed{-1 - \frac{\pi}{4}}$