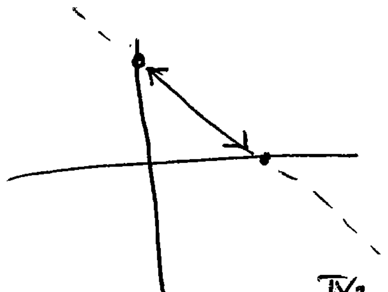
1. A particle has position given by $x(t) = \sin^2 t,\ \ y(t) = \cos^2 t$ as t varies from 0 to 3π. Find the distance traveled, and compare with the length of the curve.

Note $\sin^2 t + \cos^2 t = 1$ so curve lies on the line $x+y=1$.

It oscillates between $(0,1)$ and $(1,0)$.

$t = 0, \pi, 2\pi, \ldots$ $\quad$ $t = \pi/2, 3\pi/2, 5\pi/2, \ldots$



Length $= \int_0^{\pi/2} ds$, traversed 6 times from 0 to 3π.

$x'(t) = 2\sin t\cos t \qquad y'(t) = -2\sin t\cos t$

$ds = \sqrt{4\sin^2 t\cos^2 t + 4\sin^2 t\cos^2 t}\, dt$

$= 2\sqrt{2}\,|\sin t\cos t|\, dt \quad = 2\sqrt{2}\sin t\cos t\, dt$

since $0 \le t \le \pi/2$

$L = \int_0^{\pi/2} 2\sqrt{2}\sin t\cos t\, dt \qquad u = \sin t,\ du = \cos t\, dt$

$= \int_0^1 2\sqrt{2}\, u\, du = \sqrt{2}\, u^2 \Big|_0^1 = \underline{\sqrt{2}}.$

Dist. traveled $= \underline{6\sqrt{2}}$.

2. Consider the parametric curve $x(t) = 2t^3 + 3t^2 - 12t$, $\quad y(t) = 2t^3 + 3t^2 + 1$.

(a) Find all points on the curve where the tangent line is vertical or horizontal (and indicate which is which).

(b) Find $\dfrac{d^2y}{dx^2}$.

(a) $x'(t) = 6t^2 + 6t - 12$, $\quad y'(t) = 6t^2 + 6t$

$= 6(t+2)(t-1)$ $\quad\quad = 6t(t+1)$

horizontal when $x' \neq 0, y' = 0$

i.e. $t = 0, -1$

vertical when $x' = 0, \; y' \neq 0$

i.e. $t = -2, 1$. Now plug into $x(t), y(t)$:

Horizontal: $(0,1)$, $(13,2)$

Vertical: $(20,-3)$, $(-7,6)$

(b) $\dfrac{dy}{dx} = \dfrac{t(t+1)}{(t+2)(t-1)} = \dfrac{t^2+t}{t^2+t-2}$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{x'(t)} = \frac{\dfrac{(t^2+t-2)(2t+1)-(t^2+t)(2t+1)}{(t^2+t-2)^2}}{6(t+2)(t-1)}$$

$$= \boxed{\frac{-2(2t+1)}{6(t^2+t-2)^3}}$$

3. For each of the sequences below, determine whether it converges (and find the limit) or diverges (and whether it goes to ∞ or $-\infty$ or neither). Give brief explanations.

(a) $a_n = \dfrac{\cos^2 n}{2^n}$ $\boxed{\text{Converges to } 0}$

Since numerator is between -1 and 1

and denominator goes to $+\infty$.

(b) $b_n = \dfrac{3+5n}{n+n^2}$ Note $0 \le \dfrac{3+5n}{n+n^2} \le \dfrac{3n+5n}{n+n^2} = \dfrac{8}{1+n}$.

Since $\dfrac{8}{1+n} \to 0$, our sequence $\boxed{\text{converges to } 0}$

by the Squeeze Theorem.

Alternatively, divide top and bottom by n^2:

$$b_n = \frac{3/n^2 + 5/n}{1/n + 1} \longrightarrow \frac{0+0}{0+1} = 0.$$

(c) $c_n = n\cos(n\pi)$ $= \begin{cases} n & \text{if } n \text{ even} \\ -n & \text{if } n \text{ odd} \end{cases}$ (since $\cos(n\pi) = \pm 1$)

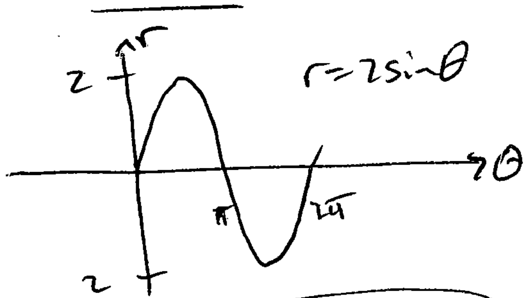
$= (-1)^n n$

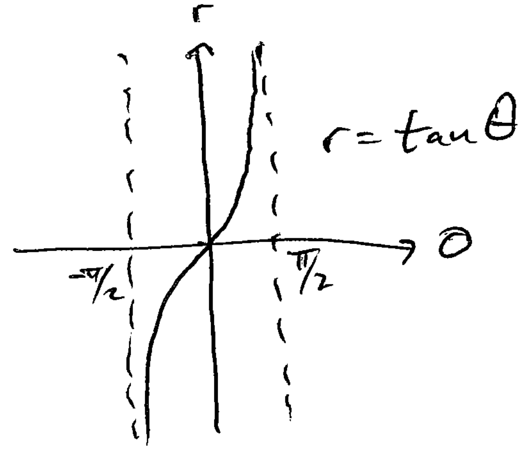
So sequence is $(-1, 2, -3, 4, -5, 6, \cdots)$

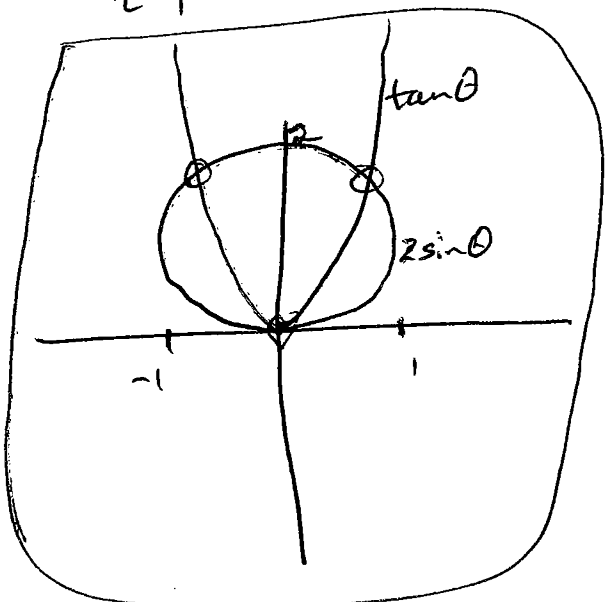
This $\boxed{\text{diverges}}$, and does not go to ∞ or $-\infty$.

4. Find all intersection points between the curves $r = 2\sin\theta$ and $r = \tan\theta$. Then draw both curves, labeling the intersection points and any other interesting features.

Curves







Both curves go through the origin. Now set

$$2\sin\theta = \tan\theta$$

$$2\sin\theta\cos\theta = \sin\theta$$

$$\sin\theta(2\cos\theta - 1) = 0$$

so $\sin\theta = 0$ or $2\cos\theta - 1 = 0$

$\theta = 0, \pi, 2\pi, \ldots$
get $r = 0$, the origin

$\cos\theta = \frac{1}{2}$

$\theta = \pi/3, 5\pi/3, 7\pi/3, 11\pi/3, \ldots$

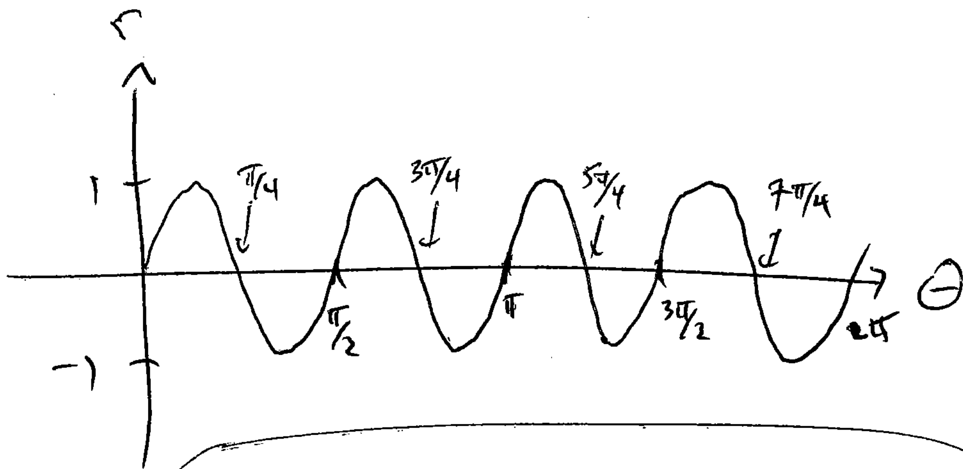
get $\begin{cases} r = \sqrt{3}, & \theta = \pi/3 \\ r = -\sqrt{3}, & \theta = 5\pi/3 \end{cases}$

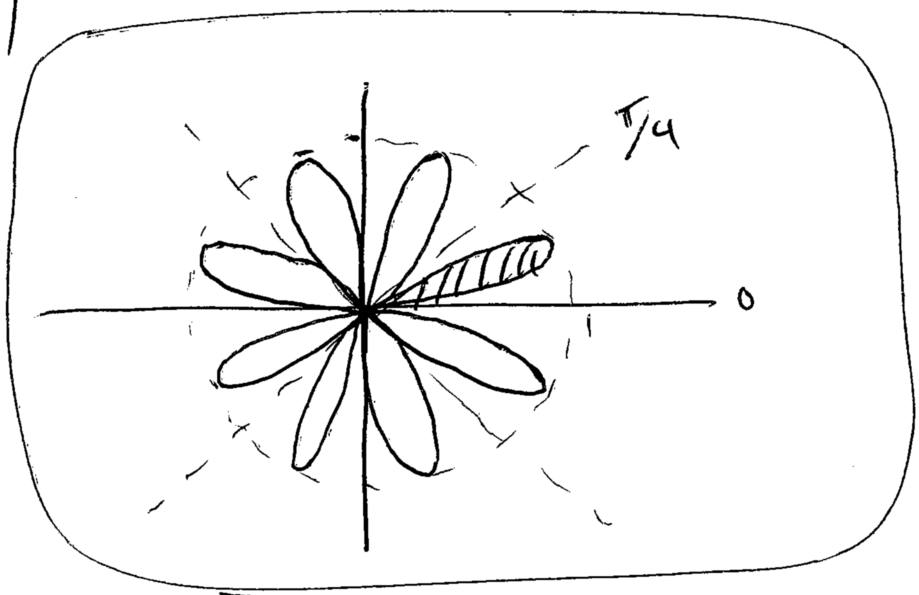
$\boxed{\text{Int. points: } (0,0),\ \left(\frac{\sqrt{3}}{2}, \frac{3}{2}\right),\ \left(\frac{-\sqrt{3}}{2}, \frac{3}{2}\right)}$

In Cartesian coords.

$(\sqrt{3}\cos(\pi/3), \sqrt{3}\sin(\pi/3))$ $\quad$ $(\sqrt{3}\cos(5\pi/3), -\sqrt{3}\sin(5\pi/3))$

5. Carefully draw the curve $r = \sin 4\theta$. Then find the area of one lobe of this curve.





$$\text{Area} = \int_0^{\pi/4} \frac{1}{2} \sin^2 4\theta \, d\theta$$

$$= \int_0^{\pi/4} \frac{1}{4} (1 - \cos 8\theta) \, d\theta$$

$$= \left[\frac{\theta}{4} - \frac{1}{32} \sin 8\theta \right]_0^{\pi/4}$$

$$= \boxed{\frac{\pi}{16}}$$

6(a) Identify the curve by finding a Cartesian equation for the curve $r = \tan\theta \sec\theta$.

$$r = \frac{\sin\theta}{\cos\theta} \cdot \frac{1}{\cos\theta}$$

$$r\cos^2\theta = \sin\theta$$

$$r^2\cos^2\theta = r\sin\theta$$

$$\boxed{x^2 = y}$$

6(b) Say carefully what it means for a sequence $\{a_n\}$ to be *bounded*, and to be *monotonic*.

Bounded: there are numbers m, M such that every a_n is between m and M.

Monotonic means Increasing or Decreasing,

Increasing: $a_{n+1} > a_n$ for every n

Decreasing: $a_{n+1} < a_n$ for every n

Extra credit (+2) Find the limit of the sequence given by $a_1 = 2$, $a_{n+1} = \dfrac{1}{3 - a_n}$.

Assume the limit exists, and is L.

Take limit of both sides: get $L = \frac{1}{3-L}$

So $3L - L^2 = 1$, or $L^2 - 3L + 1 = 0$

$\Rightarrow L = \frac{3 \pm \sqrt{5}}{2}$.

Also, can check sequence is decreasing. So $L \le 2$.

$\Rightarrow \boxed{L = \frac{3-\sqrt{5}}{2}}$