

1(a) Find a power series representation of $\frac{1}{1-x}$ centered at $a = 5$, and then find one centered at any point $a \neq 1$. [Hint: Start by writing $\frac{1}{1-x} = \frac{1}{(1-a)-(x-a)}$.]

$$\underline{a=5} \quad \frac{1}{1-x} = \frac{1}{-4-(x-5)} = -\frac{1}{4} \cdot \frac{1}{1-\left(\frac{x-5}{-4}\right)}$$

$$= -\frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{x-5}{-4}\right)^n$$

general "a"

$$= \sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^{n+1} (x-5)^n$$

$$\frac{1}{1-x} = \frac{1}{(1-a)-(x-a)} = \frac{1}{1-a} \cdot \frac{1}{1-\left(\frac{x-a}{1-a}\right)}$$

$$= \frac{1}{1-a} \sum_{n=0}^{\infty} \left(\frac{x-a}{1-a}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{(1-a)^{n+1}} (x-a)^n$$

(b) Find the intervals of convergence. What do you notice about these intervals for various values of a ? Why do they make sense?

Geometric Series converge when $|r| < 1 \rightarrow \text{d.e.}$

$$\left|\frac{x-a}{1-a}\right| < 1$$

$$|x-a| < |1-a|$$

If $a \geq 1$: $|x-a| < a-1$, d.e.

$$1 < x < 2a-1$$

or

If $a < 1$: $|x-a| < 1-a$, d.e.

$$2a-1 < x < 1$$

Notice there is always an endpoint at 1, where $\frac{1}{1-x}$ is undefined.

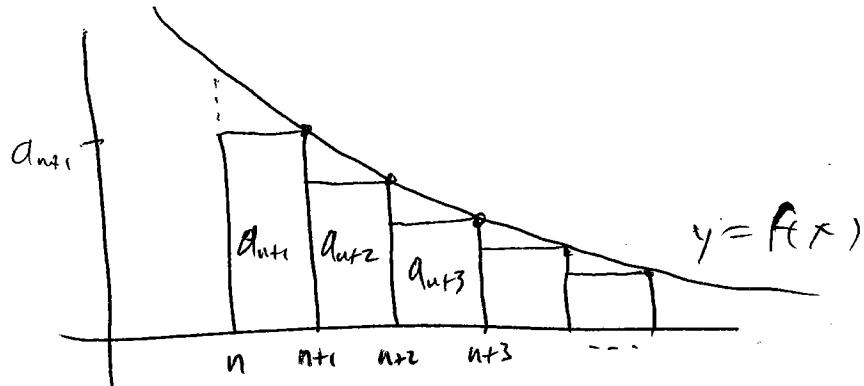
2(a) What is the remainder R_n of a convergent series $\sum_{n=0}^{\infty} a_n$? What is s_n ?

$$\boxed{S_n = a_0 + a_1 + \dots + a_n}$$

$$R_n = \left(\sum_{n=0}^{\infty} a_n \right) - S_n = a_{n+1} + a_{n+2} + \dots$$

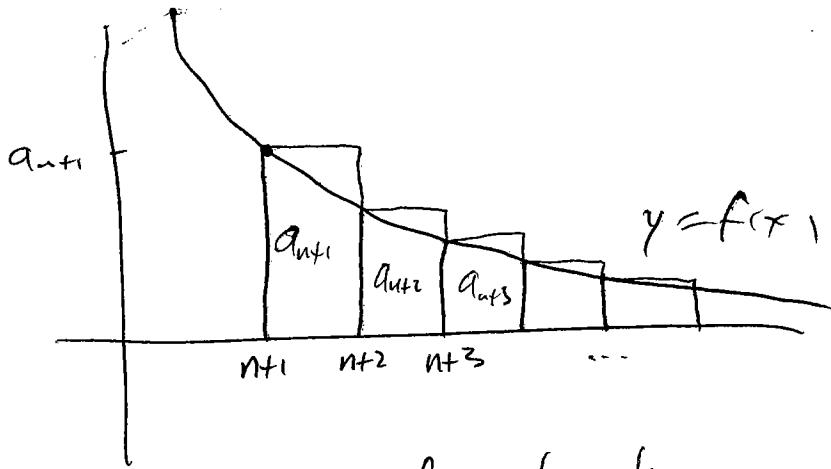
(b) Suppose $a_n = f(n)$ where $f(x)$ is continuous, positive, and decreasing. Suppose also that $\sum_{n=0}^{\infty} a_n$ converges. On the pictures below, add rectangles and any other information needed, and use them to explain how to estimate R_n from above and below by integrals. Be sure to write what the integrals are.

(i) $R_n \leq \int f(x) dx$:



$R_n = \text{area of rectangles}$
 $\leq \text{area under curve from } n \text{ to } \infty$

(ii) $\int f(x) dx \leq R_n$:



$R_n = \text{area of rectangles}$
 $\geq \text{area under curve}$
 $\text{from } nt_1 \text{ to } \infty$

3. Find the sums of the following series:

$$\begin{aligned}
 \text{(a)} \sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n^2 + 6n + 9} \right) &= \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{(n+3)^2} \\
 &= \underbrace{\left(\frac{1}{1^2} - \frac{1}{4^2} \right) + \left(\frac{1}{2^2} - \frac{1}{5^2} \right) + \left(\frac{1}{3^2} - \frac{1}{6^2} \right) + \left(\frac{1}{4^2} - \frac{1}{7^2} \right) + \left(\frac{1}{5^2} - \frac{1}{8^2} \right)}_{\text{+ ...}} \\
 &\quad \text{All terms cancel except:} \\
 &= \boxed{\frac{1}{1} + \frac{1}{4} + \frac{1}{9}}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1} (2n+1)!} &= \sum_{n=0}^{\infty} (-1)^n \left(\frac{\pi}{4} \right)^{2n+1} \frac{1}{(2n+1)!} \\
 &= \sin \left(\frac{\pi}{4} \right) \\
 &= \boxed{\frac{\sqrt{2}}{2}}
 \end{aligned}$$

$$\text{(c)} 3 - \frac{9}{2!} + \frac{27}{3!} - \frac{81}{4!} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{3^n}{n!}$$

$$\text{note } e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad , \quad e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

So our series is missing the first term. $-e^{-x}$ with $x=3$, but

$$= \boxed{1 - e^{-3}}$$

4. How many terms of the series $\sum_{n=1}^{\infty} (-1)^n \frac{\sin^2(n)}{n^3}$ should be added to get an answer within $\frac{1}{1000}$ of the actual sum? Explain briefly.

Since the series alternates, want $a_1 + \dots + a_n$
where $|a_{n+1}| \leq \frac{1}{1000}$.

Since $0 \leq \sin^2(n) \leq 1$, we have

$$|a_{10}| = \left| \frac{\sin^2(10)}{10^3} \right| \leq \frac{1}{1000}$$

So $\boxed{a_1 + \dots + a_9}$ works.

5. Evaluate the integrals as infinite series:

$$(a) \int x \cos(x^3) dx$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$x \cos(x^3) = x \left(1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \dots \right)$$

$$= x - \frac{x^7}{2!} + \frac{x^{13}}{4!} - \frac{x^{19}}{6!} + \dots$$

$$\int x \cos(x^3) dx = \frac{x^2}{2} - \frac{x^8}{8 \cdot 2!} + \frac{x^{14}}{14 \cdot 4!} - \frac{x^{20}}{20 \cdot 6!} + \dots + C$$

$$= \boxed{C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!}}$$

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n$$

$$\int \frac{1}{1+x^2} dx = \int \sum_{n=0}^{\infty} (-x^2)^n dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx$$

$$= \boxed{C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}$$

6. Test the following series for convergence or divergence. If possible, say whether it is absolutely convergent or conditionally convergent.

$$(a) \sum_{n=1}^{\infty} (-1)^{n-1} n^{-3} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^3}$$

The series $\sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{1}{n^3} \right| = \sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series with $p = 3 > 1$ so it converges.
 \Rightarrow original series converges absolutely.

$$(b) \sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$$

Compare with $\frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$

$$\frac{\sqrt{n}}{n-1} > \frac{\sqrt{n}}{n} \text{ for all } n$$

and $\sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/2}}$ diverges
 (p-series, $p = \frac{1}{2} \leq 1$)

\Rightarrow series diverges.

$$(c) \sum_{n=1}^{\infty} (\sqrt[n]{2} - 1)^n$$

root test: $\lim_{n \rightarrow \infty} \left(|(\sqrt[n]{2} - 1)^n| \right)^{1/n}$

$$= \lim_{n \rightarrow \infty} (2^{1/n} - 1) = 1 - 1 = 0$$

since limit is < 1 ,

the series converges absolutely.

7. If $\sum_{n=0}^{\infty} c_n(x-3)^n$ converges at $x = 1$ and diverges when $x = 5$, what can you say about the interval of convergence?

Interval is centered at 3, contains 1, does not contain 5

$$\Rightarrow \text{must be } [1, 5)$$

8. Find the following Taylor series:

(a) $f(x) = \frac{1}{\sqrt{x}}$ at $a = 9$

$$f'(x) = -\frac{1}{2} x^{-3/2}$$

$$f''(x) = \frac{1}{2} \cdot \frac{3}{2} x^{-5/2}$$

$$f'''(x) = -\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} x^{-7/2}$$

$$f^{(n)}(x) = (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} x^{-\frac{2n-1}{2}}$$

$$f^{(n)}(9) = (-1)^n \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n 3^{2n+1}}$$

(b) $g(x) = x^2 e^{-x}$ at $a = 0$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$$

$$\text{so } x^2 e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+2}}{n!}$$

(c) $h(x) = \ln(1-x)$ at $a = 0$

$$= - \int \frac{1}{1-x} dx$$

$$= C + - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \quad \text{Plugin } x=0 \text{ to get } C=0.$$

$$\ln(1-x) = - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$