

Name:

Id:

Final Exam
Math 2433-001
December 15, 2006

choose seven problems

Problem 1:

Problem 4:

Problem 7:

Problem 2:

Problem 5:

Problem 8:

Problem 3:

Problem 6:

Problem 9:

Total:

1. (a) Find the interval of convergence for the power series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$.

Ratio test: $\lim_{n \rightarrow \infty} \left| \frac{(x-2)^{n+1}}{n+1} \cdot \frac{n}{(x-2)^n} \right| = \lim_{n \rightarrow \infty} \left| (x-2) \frac{n}{n+1} \right| = |x-2|$

converges for $|x-2| < 1$, diverges for $|x-2| > 1$.
 \uparrow
 $1 < x < 3$

Endpoints: $x=3: \sum_{n=1}^{\infty} \frac{1}{n}$ diverges

$x=1: \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{n}\right)$

$\sum_n \frac{1}{n}$ decreases and approaches 0,
 Alt. Series Test \Rightarrow converges.

Interval:
$$\boxed{1 \leq x < 3}$$

(b) Suppose that the series $\sum_{n=0}^{\infty} c_n 4^n$ converges and $\sum_{n=0}^{\infty} c_n (-5)^n$ diverges, for the same constants c_n . What can be said about convergence or divergence of the following series?

(i) $\sum_{n=0}^{\infty} c_n 6^n$

The series $\sum_{n=0}^{\infty} c_n x^n$ has interval of convergence $-R \leq x \leq R$
 and R is between 4 and 5.

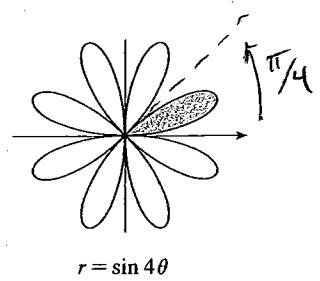
(ii) $\sum_{n=0}^{\infty} (-1)^n c_n 3^n$

$$\sum_{n=0}^{\infty} c_n (-3)^n$$

So (i) diverges and
 (ii) converges

2. (a) Find the area of one "lobe" of the curve $r = \sin 4\theta$:

$$\begin{aligned}
 A &= \int_0^{\pi/4} \frac{1}{2} (\sin 4\theta)^2 d\theta \\
 &= \int_0^{\pi/4} \frac{1}{4} (1 - \cos 8\theta) d\theta \\
 &= \frac{1}{4} \int_0^{\pi/4} d\theta - \frac{1}{4} \int_0^{\pi/4} \cos 8\theta d\theta \\
 &= \frac{\pi}{16} - \frac{1}{4} [\sin 8\theta]_0^{\pi/4} \\
 &= \boxed{\frac{\pi}{16}}
 \end{aligned}$$

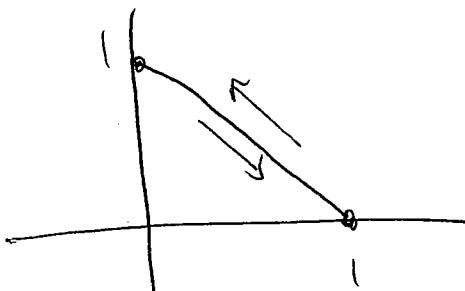


(b) Express the curve $x = \sin^2 t$, $y = \cos^2 t$ in terms of x and y only. Then draw this curve and describe carefully how it is traced as the parameter increases.

$$\sin^2 t + \cos^2 t = 1$$

$$\text{so } \boxed{x + y = 1}$$

Also, x and y are
70.



The point moves back
and forth along the line segment.

3. (a) Find a vector function that represents the curve of intersection of the surface $x = y^2$ and the plane $y - z = 4$. [Draw a picture if it helps.]

Try $y = t$ and $x = t^2$. Then $z = y - 4$
 $= t - 4$

$$\underline{\gamma}(t) = \langle t^2, t, t - 4 \rangle$$

- (b) Find a tangent vector to this curve at the point $(0, 0, -4)$.

$$\underline{\gamma}'(t) = \langle 2t, 1, 1 \rangle$$

At $(0, 0, -4)$ we have $t = 0$.

So $\underline{\gamma}'(0) = \boxed{\langle 0, 1, 1 \rangle}$ is a
 tangent vector at this point.

4. (a) Write the equation $z = x^2 + y^2$ in spherical and cylindrical coordinates.

Spherical: $\rho \cos \phi = \rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta$

$$\rho \cos \phi = \rho^2 \sin^2 \phi$$

$$\cos \phi = \rho \sin^2 \phi$$

Cylindrical: $z = r^2 \cos^2 \theta + r^2 \sin^2 \theta$

$$z = r^2$$

- (b) Identify the surface $\rho^2(\sin^2 \phi \cos^2 \theta + \cos^2 \phi) = 4$. Be as specific as you can.

$$\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \cos^2 \phi = 4$$

$$x^2 + z^2 = 4$$

Cylinder of radius 2 enclosing and centered along the y -axis,

5. Let $\mathbf{r}(t) = 3 \sin t \mathbf{i} + 4t \mathbf{j} + 3 \cos t \mathbf{k}$. Find the arc length function $s(t)$, measured from the point $t = 0$. Then parametrize the curve in terms of arc length.

$$\underline{\sigma}'(t) = \langle 3 \cos t, 4, -3 \sin t \rangle$$

$$\begin{aligned} |\underline{\sigma}'(t)|^2 &= 9 \cos^2 t + 16 + 9 \sin^2 t \\ &= 25 \end{aligned}$$

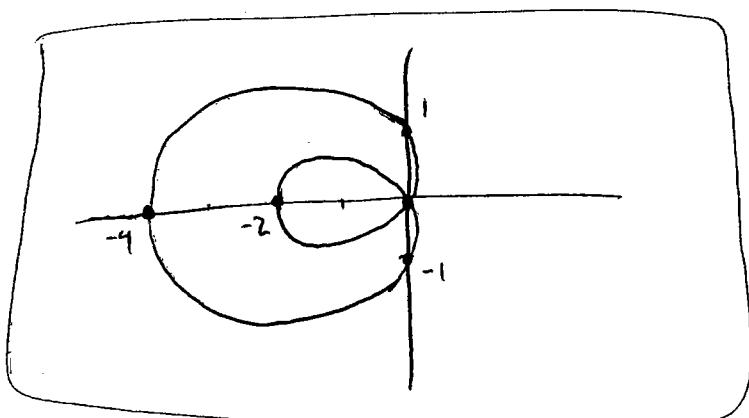
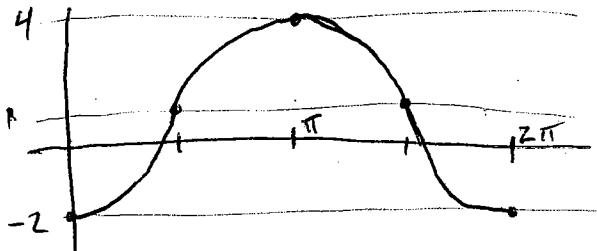
$$|\underline{\sigma}(t)| = 5$$

$$s(t) = \int_0^t 5 \, du = \boxed{5t}$$

$$\text{Thus } t = \frac{s}{5}, s \geq 0$$

$$\boxed{\underline{\sigma}(s) = \left\langle 3 \sin\left(\frac{s}{5}\right), \frac{4s}{5}, 3 \cos\left(\frac{s}{5}\right) \right\rangle}$$

6. (a) Carefully draw the curve given in polar coordinates by $r = 1 - 3 \cos \theta$.



- (b) Find an equation for the tangent line to this curve at the point $(0, 1)$.

$$x = (1 - 3 \cos \theta) \cos \theta = \cos \theta - 3 \cos^2 \theta$$

$$y = (1 - 3 \cos \theta) \sin \theta = \sin \theta - 3 \cos \theta \sin \theta$$

$$\frac{dy}{d\theta} = \cos \theta - 3(\cos^2 \theta - \sin^2 \theta)$$

$$\frac{dx}{d\theta} = -\sin \theta - 6 \cos \theta (-\sin \theta)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos \theta - 3(\cos^2 \theta - \sin^2 \theta)}{6 \cos \theta \sin \theta - \sin \theta}$$

$$\left. \frac{dy}{dx} \right|_{\theta=\pi/2} = \frac{3}{-1} = -3.$$

Tangent line: $y = -3x + 1$

7. For each series determine whether it converges absolutely, converges conditionally, or diverges. Include brief explanations.

$$(a) \sum_{n=1}^{\infty} \frac{(2n)^n}{n^{2n}}$$

Root Test:

$$\lim_{n \rightarrow \infty} (|a_n|)^{1/n} = \lim_{n \rightarrow \infty} \frac{2^n}{n^2}$$

$$= \lim_{n \rightarrow \infty} \frac{2^n}{n^2} = 0. \quad \text{This is } < 1, \text{ so}$$

the series converges absolutely.

$$(b) \text{ The series whose terms are defined by } a_1 = 2, a_{n+1} = \frac{5n+1}{4n+3} a_n. \rightarrow \frac{a_{n+1}}{a_n} = \frac{5n+1}{4n+3}$$

Ratio Test: $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{5n+1}{4n+3} \right|$

$$= \lim_{n \rightarrow \infty} \frac{5}{4} \quad \text{by L'Hopital's rule}$$

$$= \frac{5}{4} > 1$$

so it diverges.

$$(c) \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$$

$$= \sum (-1)^n \frac{1}{n^{1/3}}$$

Since $\frac{1}{n^{1/3}} \rightarrow 0$ and decreases with n ,

Alt. Series test \Rightarrow series converges.

But $\sum \frac{1}{n^{1/3}}$ diverges since $\frac{1}{3} < 1$.

\Rightarrow The series converges conditionally.

8. (a) Find the Maclaurin series for $x^2 e^{-x}$. What is its radius of convergence?

The Maclaurin series for e^{-x} is $\sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$, converging everywhere.

So $x^2 e^{-x}$ has series

$$\boxed{\sum_{n=0}^{\infty} \frac{(-x)^{n+2}}{n!}}, \text{ converging everywhere.}$$

- (b) Evaluate $\int \cos(x^3) dx$ as an infinite series.

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\cos(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!}$$

$$\int \cos(x^3) dx = \boxed{C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{(6n+1)(2n)!}}$$

- 9 (a) Find the cosine of the angle between the planes $5x - 4y + z = 2$ and $x - y - z = 14$.

The angle is also the angle between normal vectors.

$$\underline{n}_1 = \langle 5, -4, 1 \rangle, \underline{n}_2 = \langle 1, -1, -1 \rangle.$$

$$\begin{aligned} \text{then } \cos \theta &= \frac{\underline{n}_1 \cdot \underline{n}_2}{\|\underline{n}_1\| \|\underline{n}_2\|} = \frac{5+4-1}{\sqrt{25+16+1} \cdot \sqrt{3}} \\ &= \frac{8}{\sqrt{42}} = \boxed{\frac{8}{3\sqrt{14}}} \end{aligned}$$

- (b) Find an equation of the plane through the point $(6, 0, -2)$ which contains the line $x = 4 - 2t$, $y = 3 + 5t$, $z = 7 + 4t$.

line has direction $\underline{u} = \langle -2, 5, 4 \rangle$. direction from line to $(6, 0, -2)$ is $\underline{v} = (6, 0, -2) - (4, 3, 7) = \langle 2, -3, -9 \rangle$.

$$\begin{aligned} \text{Then } \underline{u} \times \underline{v} &= \langle -45+12, 8-18, 6-10 \rangle \\ &= \langle -33, -10, -4 \rangle \end{aligned}$$

$$\text{Plane is } -33x - 10y - 4z = C$$

$$\text{Plug in } (6, 0, -2): -33(6) + 8 = C$$

$$C = -190$$

$$\boxed{-33x - 10y - 4z = -190}$$