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1. (8 points) Consider the lines $\mathbf{r} = \langle t, 2t - 8, t + 1 \rangle$ and $\mathbf{r} = \langle 1 - s, 4 + 3s, 12 + 4s \rangle$.

(a) Find the point at which the lines intersect.

$$t = 1 - s$$

$$2t - 8 = 4 + 3s$$

$$2(1-s) - 8 = 4 + 3s$$

$$-10 = 5s$$

$$s = -2$$

$$t = 3$$

$$\boxed{\text{point} = (3, -2, 4)}$$

(b) Find an equation of the plane that contains these lines.

lines have directions $\underline{u} = \langle 1, 2, 1 \rangle$

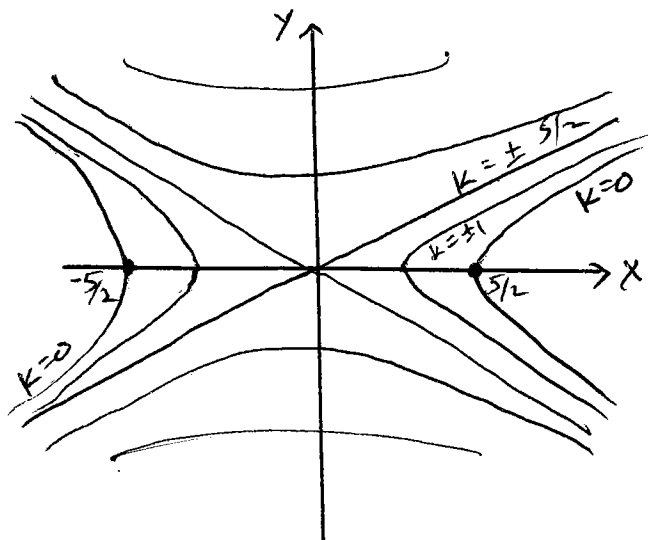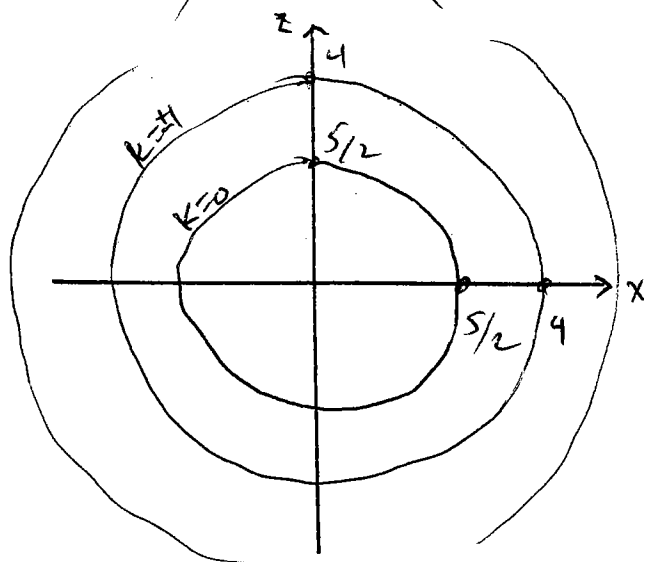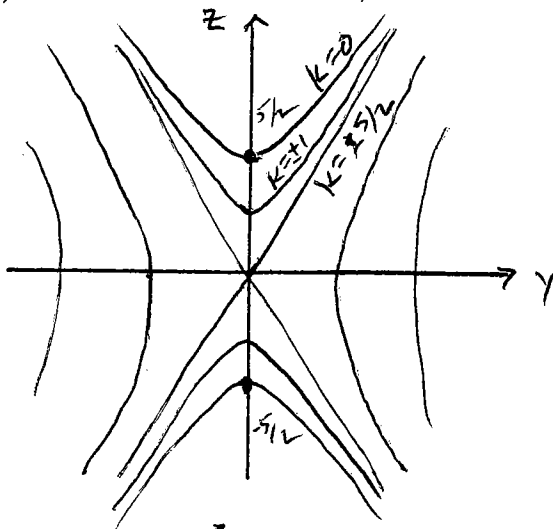and $\underline{v} = \langle -1, 3, 4 \rangle$

then $\underline{u} \times \underline{v} = \langle 5, -5, 5 \rangle$ is

perpendicular to both, hence is a normal

vector to the plane.

equation:

$$\boxed{5(x-3) - 5(y+2) + 5(z-4) = 0}$$

2. (12 points) This problem concerns the surface with equation $4x^2 - 9y^2 + 4z^2 = 25$.

(a) On each of the axes below, draw several trace curves and label them with values of k.

$\underline{x = k} \qquad 4z^2 - 9y^2 = 25 - 4k^2$

$\underline{Cases} \quad \left. \begin{array}{l} 25 - 4k^2 > 0 \\ 25 - 4k^2 < 0 \end{array} \right\} \text{hyperbolas}$

$25 - 4k^2 = 0$
then $4z^2 = 9y^2$
$z = \pm \frac{3}{2} y$
2 lines

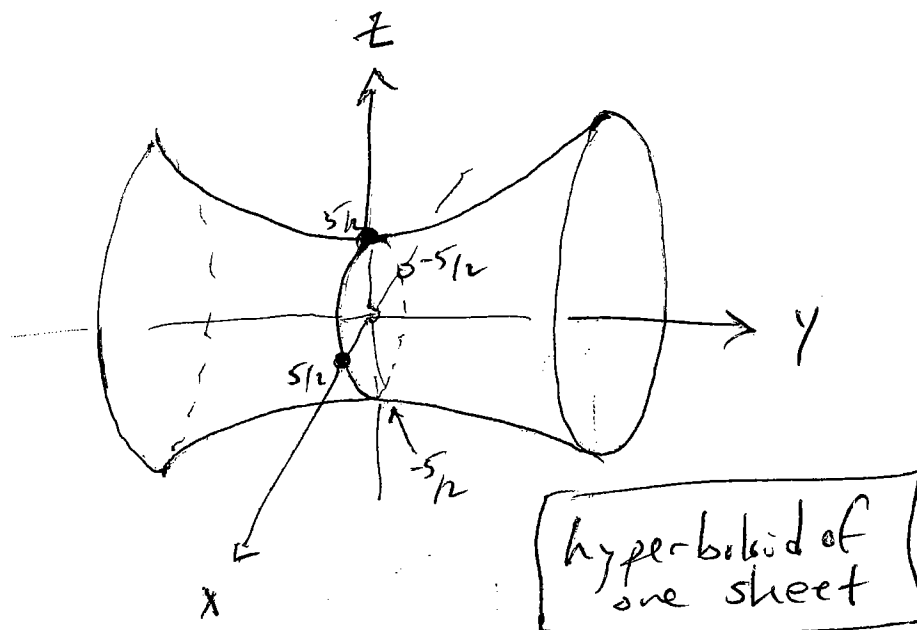$\underline{y = k} \qquad 4x^2 + 4z^2 = 25 + 9k^2$

$x^2 + z^2 = \frac{25 + 9k^2}{4}$

circles, smallest
when $k = 0$
(radius $\frac{5}{2}$)

$\underline{z = k} \qquad 4x^2 - 9y^2 = 25 - 4k^2$

similar to first
case ($x = k$)

(b) Draw the surface carefully in three dimensions. Label any intersection points with axes. What is this surface called?

hyperboloid of one sheet

3. (8 points) Find parametric equations for the line of intersection of the planes $2x + 5z + 3 = 0$ and $x - 3y + z + 2 = 0$.

the two normal vectors are

$\underline{n}_1 = \langle 2, 0, 5 \rangle$ and

$\underline{n}_2 = \langle 1, -3, 1 \rangle$. Then $\underline{n}_1 \times \underline{n}_2 = \langle 15, 3, -6 \rangle$.

Next, find a point: try $x = 0$.

$5z + 3 = 0 \implies z = -3/5$

$-3y + z + 2 = 0 \implies z + 2 = 3y$

$-3/5 + 2 = 3y$

$y = 7/15$.

$$\underline{r}(t) = \langle 15t, \ 7/15 + 3t, \ -3/5 - 6t \rangle$$

4. Let P, Q, R, and S be points such that P is not on the plane through Q, R, and S. Let $\mathbf{a} = \overrightarrow{QR}$, $\mathbf{b} = \overrightarrow{QS}$, and $\mathbf{c} = \overrightarrow{QP}$.

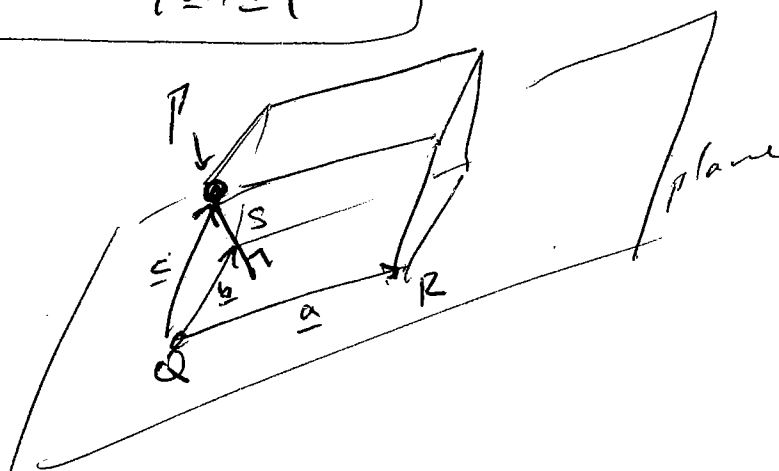**(a)** (8 points) Give formulas in terms of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ for:

(i) The area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$.

(ii) The volume of the parallelepiped spanned by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

(iii) The distance from P to the plane through Q, R, and S. [Use (i) and (ii).] Explain.

(i) $|\underline{a} \times \underline{b}|$

(ii) $|(\underline{a} \times \underline{b}) \cdot \underline{c}|$

(iii) $dist = \dfrac{|(\underline{a} \times \underline{b}) \cdot \underline{c}|}{|\underline{a} \times \underline{b}|}$

Volume = base × height
and height = distance.

(b) (4 points) Find the distance from the point $(2,1,4)$ to the plane through the points $(1,0,0)$, $(0,2,0)$, and $(0,0,3)$.

P Q

R S

$\underline{a} = \langle 1, -2, 0 \rangle$ $\underline{b} = \langle 1, 0, -3 \rangle$ $\underline{c} = \langle -1, -1, -4 \rangle$

$\underline{a} \times \underline{b} = \langle 6, 3, 2 \rangle$ $|\underline{a} \times \underline{b}| = \sqrt{36 + 9 + 4} = 7$

$(\underline{a} \times \underline{b}) \cdot \underline{c} = -6 - 3 - 8 = -17$

$|(\underline{a} \times \underline{b}) \cdot \underline{c}| = 17$

$\boxed{distance = \dfrac{17}{7}}$

5. (6 points) Find the cosine of the angle between the diagonals of two adjacent faces of a cube. (The two diagonals are chosen to meet at a point.)
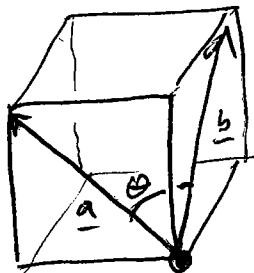
$$\underline{a} = \langle 0, -1, 1 \rangle \qquad |\underline{a}| = \sqrt{2}$$
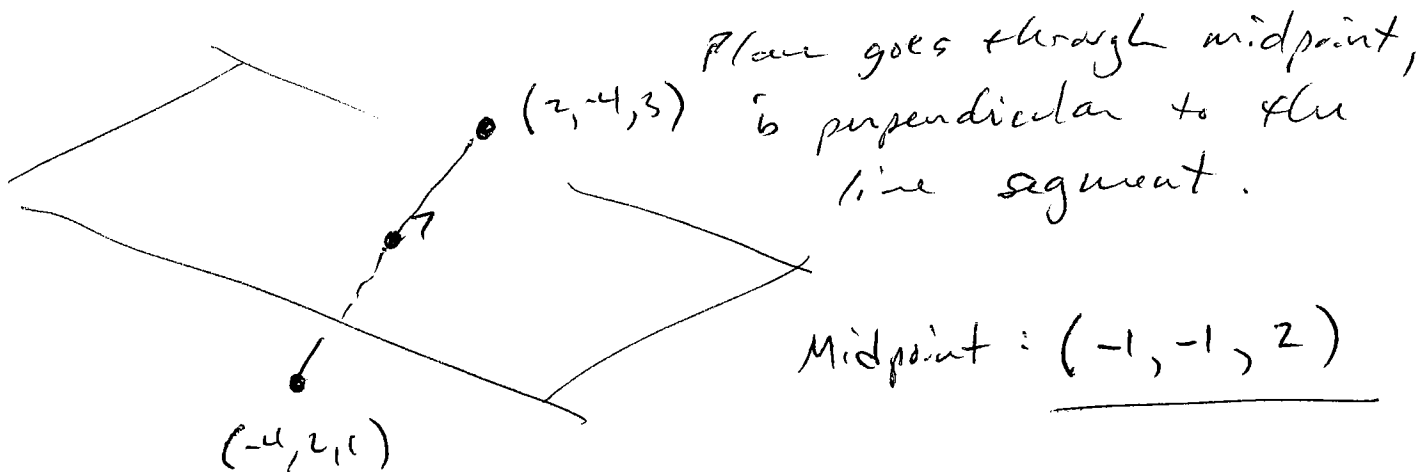
$$\underline{b} = \langle -1, 0, 1 \rangle \qquad |\underline{b}| = \sqrt{2}$$

$$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}|\cos\theta$$

$$\cos\theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}||\underline{b}|} = \frac{1}{\sqrt{2} \cdot \sqrt{2}} = \boxed{\frac{1}{2}.}$$

so $\theta = \pi/3$. (Notice — $\underline{a}$ and $\underline{b}$ are part of an equilateral triangle.)

6. (6 points) Find an equation for the plane consisting of all points that are equidistant from the points $(-4, 2, 1)$ and $(2, -4, 3)$. [Hint: draw a picture first!]
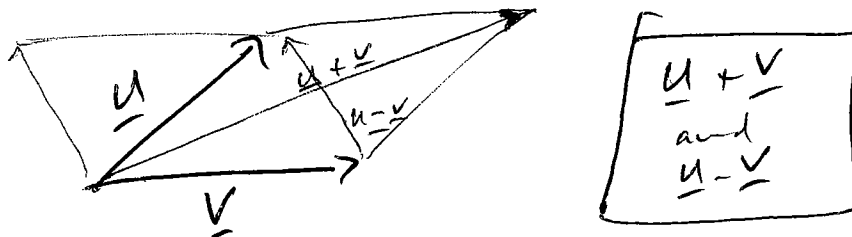
$(2, -4, 3)$

Plane goes through midpoint, is perpendicular to the line segment.

$(-4, 2, 1)$

Midpoint: $(-1, -1, 2)$

normal vector: $\langle 6, -6, 2 \rangle$

plane: $\boxed{6(x+1) - 6(y+1) + 2(z-2) = 0}$

7. (8 points) Let **u** and **v** be the sides of a parallelogram, considered as vectors.

(a) Express the diagonals of the parallelogram as vectors.

(b) Use properties of the dot product to show that if **u** and **v** have the same length, then the diagonals are perpendicular.

try this at home, for extra credit ...

(c) Use properties of the dot product to show that if the diagonals have the same length, then the parallelogram is a rectangle. [Hint: the diagonals have the same *squared* length.]

$$|\underline{u}+\underline{v}|^2 = |\underline{u}-\underline{v}|^2$$

$$\shortparallel \qquad\qquad \backslash\backslash$$

$$(\underline{u}+\underline{v})\cdot(\underline{u}+\underline{v}) \qquad (\underline{u}-\underline{v})\cdot(\underline{u}-\underline{v})$$

$$\shortparallel \qquad\qquad\qquad \backslash\backslash$$

$$(\underline{u}+\underline{v})\cdot\underline{u} + (\underline{u}+\underline{v})\cdot\underline{v} \qquad \underline{u}\cdot\underline{u}-\underline{v}\cdot\underline{u}-\underline{u}\cdot\underline{v}+\underline{v}\cdot\underline{v}$$

$$\shortparallel \qquad\qquad\qquad\qquad \shortparallel$$

$$\underline{u}\cdot\underline{u}+\underline{v}\cdot\underline{u}+\underline{u}\cdot\underline{v}+\underline{v}\cdot\underline{v} \qquad |\underline{u}|^2 - 2\underline{u}\cdot\underline{v}+|\underline{v}|^2$$

$$\shortparallel$$

$$|\underline{u}|^2 + 2\underline{u}\cdot\underline{v}+|\underline{v}|^2$$

So $2\underline{u}\cdot\underline{v} = -2\underline{u}\cdot\underline{v}$

$\Rightarrow 4\underline{u}\cdot\underline{v} = 0$

$\Rightarrow \underline{u}\cdot\underline{v} = 0$,

So it's a right angle.