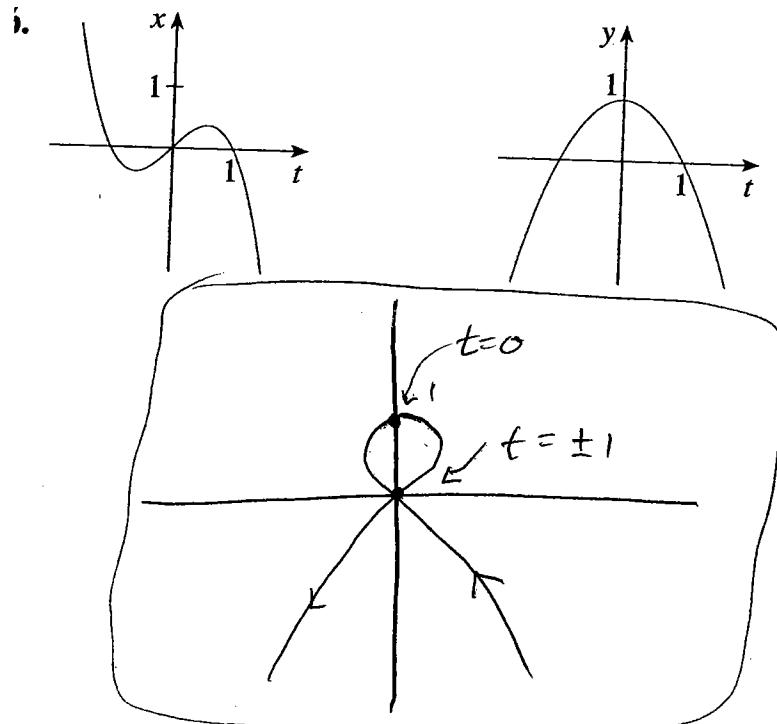
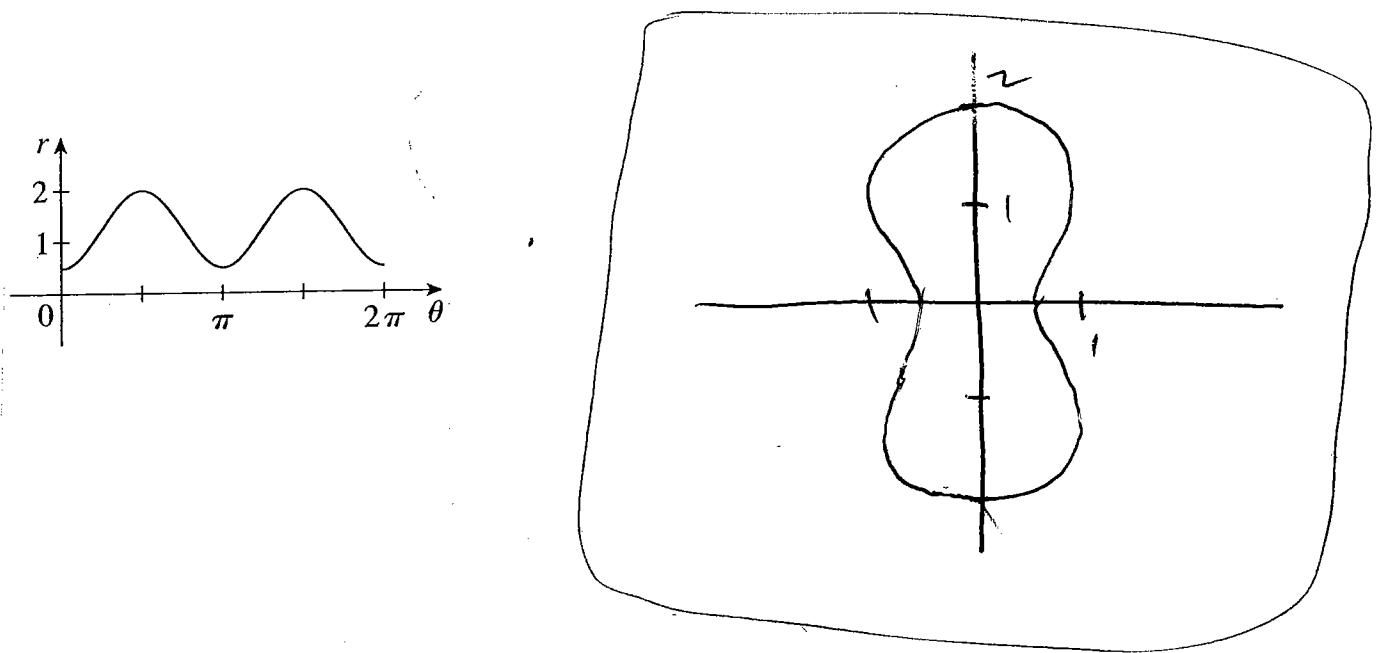


- 1(a) [4] Use the graphs of $x = f(t)$ and $y = g(t)$ to sketch the parametric curve $x = f(t)$, $y = g(t)$. Indicate with arrows the direction in which the curve is traced as t increases.



- (b) [4] The figure shows the graph of r as a function of θ in Cartesian coordinates. Use it to sketch the corresponding polar curve.



2. [6] Suppose $\{a_n\}$ is a sequence which converges, to a limit L .

(a) Explain in words why $\lim_{n \rightarrow \infty} a_{n+1} = L$.

It's the same sequence, with the first term removed.

(b) Suppose this sequence is given by $a_1 = 1$, $a_{n+1} = \frac{1}{1+a_n}$. Find L . [Hint: use part (a).]

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} a_{n+1} = \lim_{n \rightarrow \infty} \frac{1}{1+a_n} \\ &= \frac{1}{1 + \lim_{n \rightarrow \infty} a_n} = \frac{1}{1+L} \end{aligned}$$

$$\text{So } L + L^2 = 1$$

$$L^2 + L - 1 = 0$$

$$L = \frac{-1 \pm \sqrt{5}}{2}$$

Since the a_n 's are positive,

$$L = \frac{-1 + \sqrt{5}}{2}$$

3. [9] For each sequence below, find the limit or say briefly why it doesn't converge:

(a) $\sqrt{n} \cos(\frac{1}{n})$

$\cos(\frac{1}{n}) \rightarrow 1$ and \sqrt{n} increases

without bound, so it

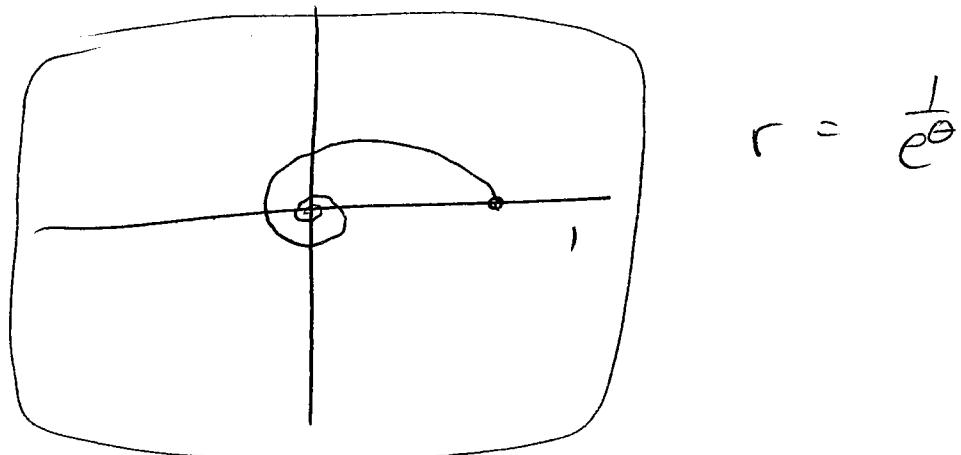
diverges

$$(b) \frac{2^n}{5^{n+4}} = \frac{1}{5^4} \left(\frac{2}{3}\right)^n$$

The limit is 0

(c) $(-1)^n \frac{n-3}{n+2}$ $\frac{n-3}{n+2} \rightarrow 1$ but there
sign alternates. It diverges.

4(a) [3] Sketch the infinite spiral $r = e^{-\theta}$, $0 \leq \theta < \infty$.



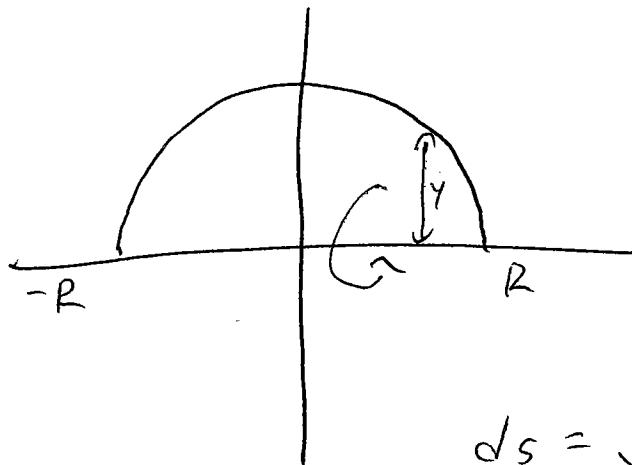
(b) [5] Find the total length of this spiral (using an improper integral).

$$\text{Length} = \int_0^\infty ds = \int_0^\infty \sqrt{e^{2\theta} + e^{2\theta}} d\theta \\ = \sqrt{2} \int_0^\infty e^{-\theta} d\theta$$

$$= \lim_{b \rightarrow \infty} \sqrt{2} \left[-e^{-\theta} \right]_0^b$$

$$= \lim_{b \rightarrow \infty} \sqrt{2} \left[-e^{-b} + e^0 \right] = \boxed{\sqrt{2}}$$

5. [8] Find the surface area of the sphere of radius R , by rotating the appropriate portion of the parametric curve $x = R \cos t$, $y = R \sin t$, about the x -axis.



$$0 \leq \theta \leq \pi$$

$$\begin{aligned}x'(t) &= -R \sin t \\y'(t) &= R \cos t\end{aligned}$$

$$\begin{aligned}ds &= \sqrt{R^2 \sin^2 t + R^2 \cos^2 t} dt \\&= R dt\end{aligned}$$

$$SA = \int_0^\pi 2\pi y \, ds$$

$$= \int_0^\pi 2\pi R \sin t \, R dt$$

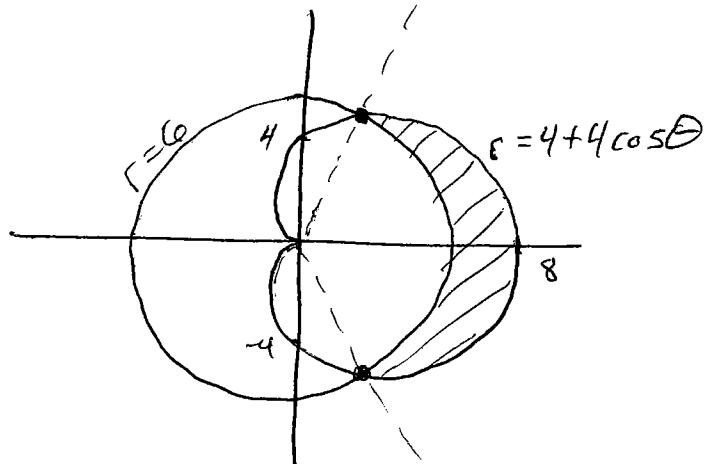
$$= 2\pi R^2 \int_0^\pi \sin t \, dt$$

$$= 2\pi R^2 \left[-\cos t \right]_0^\pi$$

$$= 2\pi R^2 [1 + 1]$$

$$= \boxed{4\pi R^2}$$

6. [8] Find the area of the region that is inside the cardioid $r = 4 + 4\cos\theta$ and outside the circle $r = 6$ (draw a picture first!).



Intersection points:

$$4 + 4\cos\theta = 6$$

$$4\cos\theta = 2$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \pm \frac{\pi}{3}$$

$$\text{Area} = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left[\frac{1}{2}(4+4\cos\theta)^2 - \frac{1}{2}(6)^2 \right] d\theta$$

$$= \frac{1}{2} \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} [16 + 32\cos\theta + 16\cos^2\theta - 36] d\theta$$

$$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} [16\cos\theta + 4(1 + \cos 2\theta) - 10] d\theta$$

$$= \left[16\sin\theta + 4\theta + 2\sin 2\theta - 10\theta \right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= 16\frac{\sqrt{3}}{2} + \frac{4\pi}{3} + 2\frac{\sqrt{3}}{2} - 10\frac{\pi}{3}$$

$$+ 16\frac{\sqrt{3}}{2} + \frac{4\pi}{3} + 2\frac{\sqrt{3}}{2} - 10\frac{\pi}{3}$$

$$= \boxed{18\sqrt{3} - 4\pi}$$

7(a) [4] True or False:

- (i) If
- $\{a_n\}$
- and
- $\{b_n\}$
- converge then
- $\{a_n + b_n\}$
- converges.

true

- (ii) If
- $\{a_n\}$
- and
- $\{b_n\}$
- are monotonic then
- $\{a_n + b_n\}$
- is monotonic.

false

7(b) [4] Give an example of:

- (i) A bounded sequence that does not converge.

$$\{(-1)^n\} = (-1, 1, -1, 1, \dots)$$

- (ii) A convergent sequence that is not monotonic.

$$\left\{\frac{(-1)^n}{n}\right\} = \left(-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, \dots\right) \rightarrow 0$$

8. [5] Suppose a bee follows the trajectory $x = 2 - 2 \sin t$, $y = t - \cos t$. $0 \leq t \leq 8$. At what times is the bee flying horizontally?

$$\frac{dx}{dt} = -2 \cos t, \quad \frac{dy}{dt} = 1 + \sin t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 + \sin t}{-2 \cos t}$$

flight is horizontal when $1 + \sin t = 0$
 (i.e. when $\frac{dy}{dt} = 0$)

$$\text{i.e. } \sin t = -1,$$

$$\text{i.e. } t = \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \dots$$

$\left[t = \frac{3\pi}{2}\right]$ is the only one between 0 and 8.

~~Note~~ $-2 \cos t = 0$ at this time, so $\frac{dy}{dx}$ looks like $\frac{0}{0}$. The slope is then

$$\lim_{t \rightarrow \frac{3\pi}{2}} \frac{1 + \sin t}{-2 \cos t} = \lim_{t \rightarrow \frac{3\pi}{2}} \frac{\cos t}{2 \sin t} = \frac{0}{2} \quad \text{by L'Hopital's rule. So the slope is indeed 0 at } t = \frac{3\pi}{2}.$$