§7.2

10. $\displaystyle\int_0^\pi \sin^2 t\,\cos^4 t\,dt = \int_0^\pi \left(\frac{1-\cos(2t)}{2}\right)\left(\frac{1+\cos(2t)}{2}\right)^2 dt = \frac{1}{8}\int_0^\pi (1-\cos^2(2t))(1+\cos(2t))\,dt$

$\displaystyle = \frac{1}{8}\int_0^\pi (1-\cos^2(2t)+\cos(2t)-\cos^3(2t))\,dt = \frac{1}{8}\left[\int_0^\pi 1\,dt + \overset{⊛}{\int_0^\pi} \cos(2t)(\underbrace{1-\cos^2(2t)}_{\sin^2(2t)})\,dt - \int_0^\pi \frac{1+\cos(4t)}{2}\,dt\right]$

⊛ Let $u = \sin(2t)$ $\quad du = 2\cos(2t)\,dt$

$t = 0 \to u = 0 \qquad t = \pi \to u = 0$

$\displaystyle = \frac{1}{8}\left[\int_0^\pi 1\,dt + \int_0^0 u^2\left(\frac{1}{2}du\right) - \frac{1}{2}\int_0^\pi (1+\cos(4t))\,dt\right]$ 0

$\displaystyle = \frac{1}{8}\left[t\Big|_0^\pi - \frac{1}{2}\left[t\Big|_0^\pi + \int_0^\pi \cos(4t)\,dt\right]\right.$ $\qquad$ ‡ Let $u = 4t \to \frac{1}{4}du = dt$

$\displaystyle = \frac{1}{8}\left[(\pi - 0) - \frac{1}{2}\left[(\pi - 0) + \frac{1}{4}\int_0^{4\pi}\cos u\,du\right]\right]$

$\displaystyle = \frac{1}{8}\left(\frac{\pi}{2} + \frac{1}{4}\sin u\Big|_0^{4\pi}\right) = \frac{1}{8}\left(\frac{\pi}{2} + 0\right) = \underline{\frac{\pi}{16}}$

22. $\displaystyle\int \tan^2\theta \sec^4\theta\,d\theta = \int \tan^2\theta\,(1+\tan^2\theta)\sec^2\theta\,d\theta$ $\quad [1+\tan^2\theta = \sec^2\theta]$

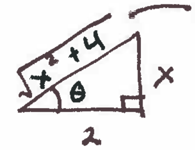
Let $u = \tan\theta \to du = \sec^2\theta\,d\theta$

$\displaystyle = \int u^2(1+u^2)\,du = \int (u^2+u^4)\,du = \frac{u^3}{3} + \frac{u^5}{5} + C$

$\displaystyle = \frac{\tan^3\theta}{3} + \frac{\tan^5\theta}{5} + C$

§7.3

2. $\displaystyle\int \frac{x^3}{\sqrt{x^2+4}}\,dx \qquad x = 2\tan\theta \Longrightarrow \frac{x}{2} = \tan\theta$



by Pythagorean Theorem.

If $x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta\,d\theta$

$\displaystyle = \int \frac{(2\tan\theta)^3}{\underbrace{\sqrt{(2\tan\theta)^2+4}}}\cdot 2\sec^2\theta\,d\theta = \int \frac{16\tan^3\theta\sec^2\theta\,d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{16\tan^3\theta\sec^2\theta\,d\theta}{2\sec\theta}$

$(2\tan\theta)^2 + 4 = 4\tan^2\theta + 4 = 4(\tan^2\theta + 1) = 4\sec^2\theta$

$\displaystyle = 8\int \tan^3\theta\sec\theta\,d\theta = 8\int \tan^2\theta\,\tan\theta\sec\theta\,d\theta = 8\int(\sec^2\theta - 1)\tan\theta\sec\theta\,d\theta$

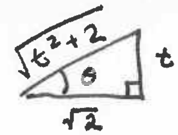
Let $u = \sec\theta \to du = \sec\theta\tan\theta\,d\theta$

$\displaystyle = 8\int (u^2-1)\,du = 8\left[\frac{u^3}{3} - u\right] + C = \frac{8u^3}{3} - 8u + C = \frac{8\sec^3\theta}{3} - 8\sec\theta + C$

By above Δ, $\quad \sec\theta = \dfrac{\sqrt{x^2+4}}{2}$

$\displaystyle = \frac{8}{3}\left(\frac{\sqrt{x^2+4}}{2}\right)^3 - 8\left(\frac{\sqrt{x^2+4}}{2}\right) + C = \frac{\left(\sqrt{x^2+4}\right)^3}{3} - 4\sqrt{x^2+4} + C = \underline{\frac{\sqrt{x^2+4}}{3}(x^2-8) + C}$

10. $\displaystyle\int \frac{t^5}{\sqrt{t^2+2}}\,dt \qquad t = \sqrt{2}\tan\theta \qquad dt = \sqrt{2}\sec^2\theta\,d\theta$



$$= \int \frac{(\sqrt{2}\tan\theta)^5}{\sqrt{(\sqrt{2}\tan\theta)^2+2}} \cdot \sqrt{2}\sec^2\theta\,d\theta = \int \frac{2^{5/2}\tan^5\theta \cdot 2^{\frac{1}{2}}\sec^2\theta\,d\theta}{\sqrt{2\sec^2\theta}}$$

$$(\sqrt{2}\tan\theta)^2 + 2 = 2\tan^2\theta + 2 = 2(\tan^2\theta + 1) = 2\sec^2\theta$$

$$= \int \frac{2^{5/2}\tan^5\theta\sec^2\theta}{\sec\theta}\,d\theta = 2^{5/2}\int \tan^5\theta\sec\theta\,d\theta = 2^{5/2}\int \tan^4\theta \cdot \tan\theta\sec\theta\,d\theta$$

$= 2^{5/2}\int (\sec^2\theta - 1)^2\tan\theta\sec\theta\,d\theta$ Let $u = \sec\theta \rightarrow du = \sec\theta\tan\theta\,d\theta$

$$= 2^{5/2}\int (u^2-1)^2\,du = 2^{5/2}\int (u^4 - 2u^2 + 1)\,du = 2^{5/2}\left(\frac{u^5}{5} - \frac{2u^3}{3} + u\right) + C$$

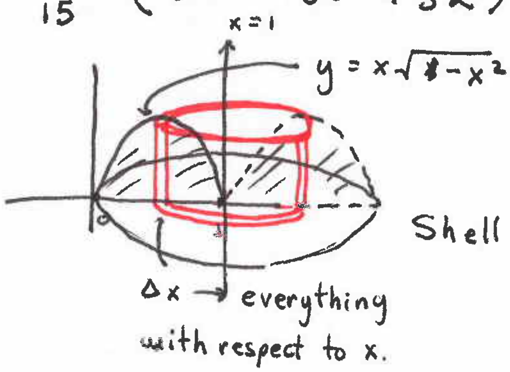
$= 2^{5/2}\left(\dfrac{\sec^5\theta}{5} - \dfrac{2\sec^3\theta}{3} + \sec\theta\right) + C$ By Δ $\sec\theta = \dfrac{\sqrt{t^2+2}}{\sqrt{2}}$

$$= 2^{5/2}\left(\frac{1}{5}\left(\frac{\sqrt{t^2+2}}{\sqrt{2}}\right)^5 - \frac{2}{3}\left(\frac{\sqrt{t^2+2}}{\sqrt{2}}\right)^3 + \left(\frac{\sqrt{t^2+2}}{\sqrt{2}}\right)\right) + C$$

$$= 4\sqrt{t^2+2}\left(\frac{1}{20}(t^2+2)^2 - \frac{2}{3}(t^2+2) + 1\right) + C$$

$$= \frac{\sqrt{t^2+2}}{15}\left(3t^4 - 8t^2 + 32\right) + C$$

38.



Rotate region below $y = x\sqrt{1-x^2}$ from $0 \le x \le 1$ about $x = 1$

Shell method: $h = y = x\sqrt{1-x^2}$

$r = 1 - x$

$$V = 2\pi\int_0^1 (1-x)(x\sqrt{1-x^2})\,dx$$

$$= 2\pi\int_0^1 x\sqrt{1-x^2}\,dx - 2\pi\int_0^1 x^2\sqrt{1-x^2}\,dx$$

① ②

① $u = 1 - x^2 \rightarrow du = -2x\,dx \qquad x = 0 \rightarrow u = 1$

$-\frac{1}{2}du = x\,dx \qquad x = 0 \rightarrow u = 1$

$$2\pi\int_0^1 x\sqrt{1-x^2}\,dx = 2\pi\int_1^0 \sqrt{u}\left(-\tfrac{1}{2}du\right) = \pi\int_0^1 u^{\frac{1}{2}}\,du = \frac{2\pi}{3}u^{3/2}\Big|_0^1 = \frac{2\pi}{3}$$

② $x = \sin\theta \quad dx = \cos\theta\,d\theta \qquad x = 0 \rightarrow \theta = 0 \qquad x = 1 \longrightarrow \theta = \frac{\pi}{2}$

$\longrightarrow$

$$2\pi\int_0^1 x^2\sqrt{1-x^2}\,dx = 2\pi\int_0^{\frac{\pi}{2}} \sin^2\theta\,\sqrt{1-\sin^2\theta}\cdot\cos\theta\,d\theta$$

$$= 2\pi\int_0^{\pi/2} \sin^2\theta\,\sqrt{\cos^2\theta}\,\cos\theta\,d\theta$$

$$= 2\pi\int_0^{\pi/2} \sin^2\theta\cos^2\theta\,d\theta = 2\pi\int_0^{\pi/2}\left(\frac{1-\cos(2\theta)}{2}\right)\left(\frac{1+\cos(2\theta)}{2}\right)d\theta$$

$$= \frac{\pi}{2}\int_0^{\frac{\pi}{2}}(1-\cos^2(2\theta))\,d\theta = \frac{\pi}{2}\int_0^{\frac{\pi}{2}}\sin^2(2\theta)\,d\theta = \frac{\pi}{4}\int_0^{\frac{\pi}{2}}(1-\cos(4\theta))\,d\theta$$

Let $u = 4\theta \quad du = 4\theta\, d\theta \rightarrow \frac{1}{4}du = d\theta \qquad \theta = 0 \rightarrow u = 0$

$\theta = \frac{\pi}{2} \rightarrow u = 2\pi$

$$= \frac{\pi}{4}\int_0^{2\pi}(1-\cos u)\frac{1}{4}\,du = \frac{\pi}{16}\int_0^{2\pi}(1-\cos u)\,du = \frac{\pi}{16}\Big[u-\sin u\Big]_0^{2\pi}$$

$$= \frac{\pi}{16}\left[(2\pi - \sin 2\pi) - (0-\sin 0)\right] = \frac{\pi}{16}\cdot 2\pi = \frac{\pi^2}{8}$$

$$\therefore\ V = 2\pi\int_0^1 (1-x)(x\sqrt{1-x^2})\,dx = \underline{\frac{2\pi}{3} - \frac{\pi^2}{8}}$$