1a. Differentiate:

(i) $f(x) = \ln(x^2 e^x)$

$$\boxed{\frac{1}{x^2e^x}\left(2xe^x + x^2e^x\right)}$$

(ii) $g(x) = (2x+1)^6(x^4-3)^5$, using logarithmic differentiation

$$y = (2x+1)^6(x^4-3)^5 \to \ln y = 6\ln(2x+1) + 5\ln(x^4-3)$$

$$\frac{1}{y}\frac{dy}{dx} = 6\cdot\frac{2}{2x+1} + 5\cdot\frac{4x^3}{x^4-3}$$

$$g'(x) = \frac{dy}{dx} = \boxed{(2x+1)^6(x^4-3)^5\left(\frac{12}{2x+1} + \frac{20x^3}{x^4-3}\right)}$$

1b. Integrate:

(i) $\int \frac{t^2}{\sqrt{1-t^6}}\,dt$

$$u = t^3, \quad du = 3t^2\,dt$$

$$= \frac{1}{3}\int\frac{du}{\sqrt{1-u^2}} = \frac{1}{3}\sin^{-1}(u) + C$$

$$= \boxed{\frac{1}{3}\sin^{-1}(t^3) + C}$$

(ii) $\int \frac{\cos(\ln x)}{x}\,dx$

$$u = \ln x \quad du = \frac{1}{x}\,dx$$

$$= \int \cos u\,du = \sin u + C$$

$$= \boxed{\sin(\ln x) + C}$$

2(a) Using the definitions of $\sinh x$ and $\cosh x$, verify the identity $\sinh 2x = 2\sinh x \cosh x$.

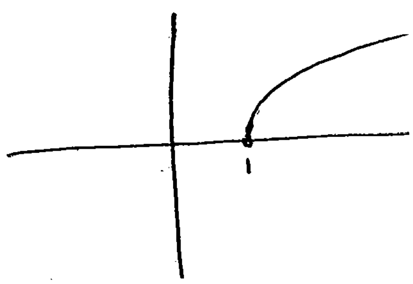
$$\sinh(2x) = \tfrac{1}{2}(e^{2x} - e^{-2x})$$

$$2\sinh x \cosh x = \tfrac{1}{2}(e^{x} - e^{-x})(e^{x} + e^{-x})$$

$$= \tfrac{1}{2}(e^{2x} - e^{-x}e^{x} + e^{x}e^{-x} - e^{-2x})$$

$$= \tfrac{1}{2}(e^{2x} - e^{-2x})$$

(b) Sketch the graph of $y = \cosh^{-1} x$. What are the domain and range of this function?



Domain: $[1, \infty)$

Range: $[0, \infty)$

(c) Evaluate

(i) $\cosh(\ln 5)$

$$\tfrac{1}{2}(e^{\ln 5} + e^{-\ln 5}) = \tfrac{1}{2}(e^{\ln 5} + e^{\ln(5^{-1})})$$

$$= \boxed{\tfrac{1}{2}\left(5 + \tfrac{1}{5}\right)}$$

(ii) $\sinh^{-1}(1)$

$$x = \sinh^{-1}(1) \iff \sinh x = 1$$

so

$$\tfrac{1}{2}(e^{x} - e^{-x}) = 1$$

$$(e^{x} - e^{-x}) = 2$$

$$e^{x} - 2 - e^{-x} = 0$$

$$e^{2x} - 2e^{x} - 1 = 0$$

$$e^{x} = \frac{2 \pm \sqrt{4+4}}{2}$$

$$e^{x} = 1 + \sqrt{2}$$

since $e^{x} > 0$

$$x = \boxed{\ln(1+\sqrt{2})}$$

3. Find the limits, and show your reasoning.

(a) $\lim_{x \to 0} \frac{\sin x}{\sinh x}$ $\to \frac{0}{0}$ indeterminate

$$= \lim_{x \to 0} \frac{\cos x}{\cosh x} = \frac{1}{1} = \boxed{1}$$

(b) $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$ $\to \frac{0}{0}$ indeterminate

$$= \lim_{x \to 0} \frac{\sin x}{2x} = \frac{1}{2} \lim_{x \to 0} \frac{\sin x}{x} = \boxed{\frac{1}{2}}$$

(or use L'Hospital again)

(c) $\lim_{x \to \pi/4} \frac{1 - \tan x}{\cos x}$

$$= \frac{0}{\sqrt{2}/2} = \boxed{0}$$

(d) $\lim_{x \to 0+} x^x$

$$y = x^x$$

$$\ln y = x \ln x$$

$$\lim_{x \to 0^+} x \ln x = 0 \cdot (-\infty) \text{ indeterminate}$$

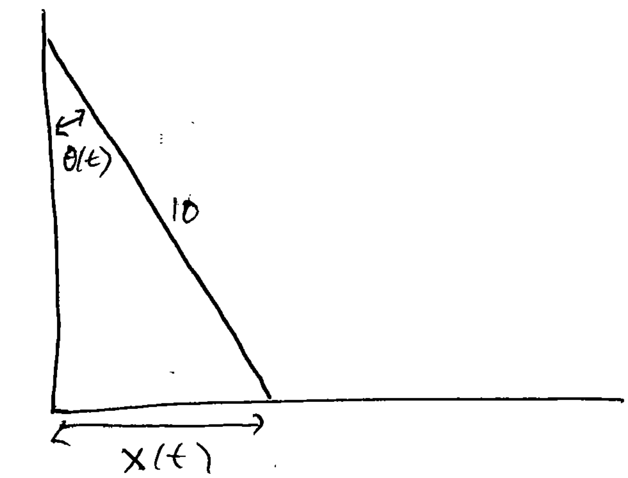
$$\lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} \frac{-x^2}{x} \to \frac{0}{0} \text{ indet.}$$

$$= \lim_{x \to 0^+} \frac{-2x}{1} = 0.$$

So $\lim_{x \to 0^+} \ln(y) = 0$

$$\Rightarrow \lim_{x \to 0^+} y = e^0 = \boxed{1}.$$

4. Solve the following related rates problem. A ladder 10 ft long leans against a vertical wall. If the bottom of the ladder slides away from the base of the wall at a speed of 8 ft/sec, how fast is the angle between the ladder and the wall changing when the bottom of the ladder is 6 ft from the base of the wall?



$$\sin\theta = \frac{x}{10}$$

$$\theta = \sin^{-1}\left(\frac{x}{10}\right)$$

diff. w.r.t. t :

$$\theta'(t) = \frac{1}{\sqrt{1-\left(\frac{x}{10}\right)^2}} \cdot \frac{1}{10}\, x'(t)$$

At the given instant,

$x(t) = 6$ and $x'(t) = 8$

So $$\theta'(t) = \frac{4}{5} \frac{1}{\sqrt{1-\left(\frac{6}{10}\right)^2}}$$

$$= \frac{4}{5} \frac{1}{\sqrt{\frac{64}{100}}} = \frac{4}{5} \frac{1}{\frac{8}{10}}$$

$$= \boxed{1 \text{ radian/sec.}}$$

5(a) Show that $\ln(x^r) = r\ln(x)$. [Hint: start by showing that the two sides differ by a constant, by differentiating both sides.]

left has derivative $\frac{1}{x^r} \cdot r x^{r-1} = \frac{r}{x}$

right has derivative $r \cdot \frac{1}{x} = \frac{r}{x}$.

$\Rightarrow$ $\ln(x^r) = r\ln(x) + C$ for some constant C.

To find C, plug something in for x, say 1:

$\ln(1^r) = r\ln(1) + C$

$0 = 0 + C \Rightarrow C = 0$

so $\ln(x^r) = r\ln(x)$ for all x.

(b) Simplify

(i) $\log_8 320 - \log_8 5$ $= \log_8\left(\frac{320}{5}\right) = \log_8(64)$

$= \log_8(8^2) = 2\log_8(8)$

$= \boxed{2}$

(ii) $10^{(\log_{10} 3 + \log_{10} 11)}$ $= 10^{\log_{10} 3} \cdot 10^{\log_{10} 11}$

$= 3 \cdot 11$

$= \boxed{33}$

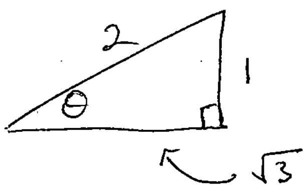
6(a) Give the domains and ranges of $f(x) = \cos^{-1} x$, $g(x) = \sin^{-1} x$, and $h(x) = \tan^{-1} x$.

	$f(x)$	$g(x)$	$h(x)$
Domain	$[-1, 1]$	$[-1, 1]$	$(-\infty, \infty)$
Range	$[0, \pi]$	$[-\pi/2, \pi/2]$	$(-\frac{\pi}{2}, \frac{\pi}{2})$

(b) Evaluate

(i) $\tan(\sin^{-1}(\frac{1}{2}))$

$\sin\theta = \frac{1}{2}$



$\tan\theta = \boxed{\frac{1}{\sqrt{3}}}$

(ii) $\cos^{-1}(\cos(\frac{7\pi}{2}))$ $= \cos^{-1}(0) = \boxed{\frac{\pi}{2}}$

(iii) $\sin(\sin^{-1}(\frac{2}{3}))$ $= \boxed{\frac{2}{3}}$

Extra credit. Find $\lim\limits_{x\to 0} \frac{1}{x^3} \int_0^x \sin(t^2)\, dt$.

$\lim\limits_{x\to 0} \frac{\int_0^x \sin(t^2)\,dt}{x^3} \to \frac{0}{0}$ indeterminate

So limit is $\lim\limits_{x\to 0} \frac{\sin(x^2)}{3x^2} \to \frac{0}{0}$ indet.

$= \lim\limits_{x\to 0} \frac{2x\cos(x^2)}{6x} = \lim\limits_{x\to 0} \frac{2\cos(x^2)}{6} = \boxed{\frac{1}{3}}$