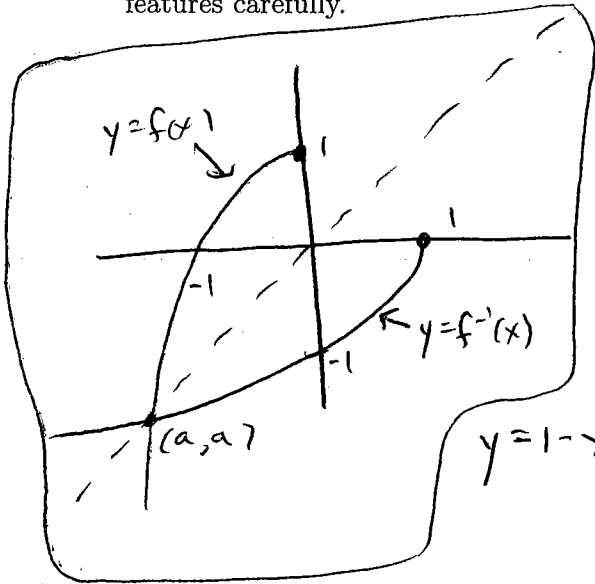


1(a) Let $f(x) = 1 - x^2$ for $x \leq 0$.

(i) What are the domain and range of $f^{-1}(x)$?

(ii) Find $f^{-1}(x)$.

(iii) Draw the graphs of $y = f^{-1}(x)$ and $y = f(x)$ (in a single picture). Label any interesting features carefully.



domain of f : $x \leq 0$

range of f : $x \leq 1$

So

domain of f^{-1} : $x \leq 1$

range of f^{-1} : $x \leq 0$

$$y = 1 - x^2 \rightarrow x^2 = 1 - y \Rightarrow x = \underbrace{-\sqrt{1-y}}_{\text{since } x \leq 0}$$

$$f^{-1}(x) = -\sqrt{1-x}$$

graphs: where "a" is a solution to $1 - x^2 = -\sqrt{1-x}$.

(b) If $g(x) = x^5 - x^3 + 2x$, find $(g^{-1})'(a)$ when $a = 2$.

$$(g^{-1})'(2) = \frac{1}{g'(g^{-1}(2))}$$

$g'(x) = 5x^4 - 3x^2 + 2$. Notice, also, that putting $x=1$ into $g(x)$ gives 2. So $g^{-1}(2) = 1$.

$$(g^{-1})'(2) = \frac{1}{g'(1)} = \boxed{\frac{1}{4}}$$

2. Consider the solid obtained by rotating about the y -axis the region in the upper right quadrant bounded by the curves $y = 4x^2 - x^4$ and $y = 0$.

(a) Find the volume of this region.

(b) You will find that only one of the two methods (washers, shells) works well. Explain what goes wrong with the other method.

(a) shells:

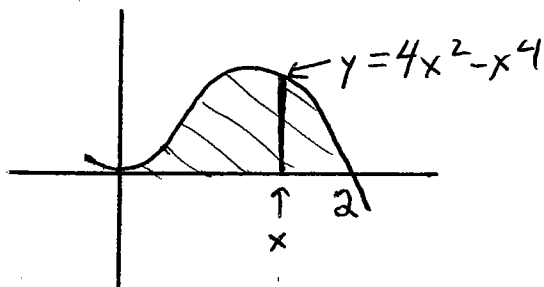


$$\begin{aligned} \text{perimeter} &= 2\pi x \\ \text{height} &= 4x^2 - x^4 \end{aligned}$$

$$\text{Area}(x) = 2\pi x (4x^2 - x^4) = 2\pi (4x^3 - x^5).$$

$$\text{Volume} = \int_0^2 2\pi (4x^3 - x^5) dx = 2\pi \left[x^4 - \frac{1}{6}x^6 \right]_0^2$$

$$= 2\pi \left(16 - \frac{64}{6} \right) = \boxed{\frac{32\pi}{3}}.$$



(b)



To find inner and outer radii of washers, we would need to solve for the two values of x in $y = 4x^2 - x^4$. It's not clear how to do this.

3(a) Set up the integral representing the area of the region bounded by the curve $y = x^2$, its tangent line at $(1, 1)$, and the x -axis. (Draw a picture.)

Tangent line: $y' = 2x, = 2$ at $x=1$.

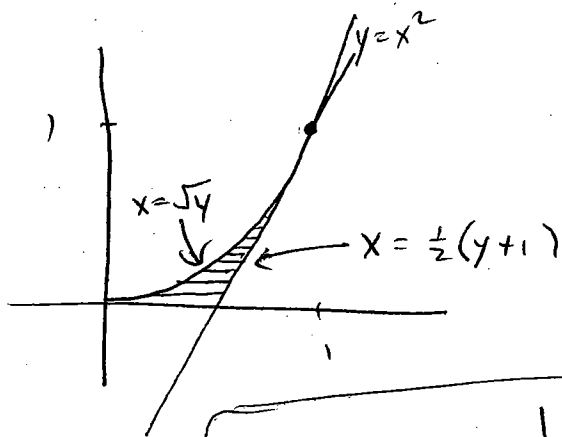
point-slope formula: $(y-1) = 2(x-1)$.

\downarrow

$$y-1 = 2x-2$$

$$2x = y+1$$

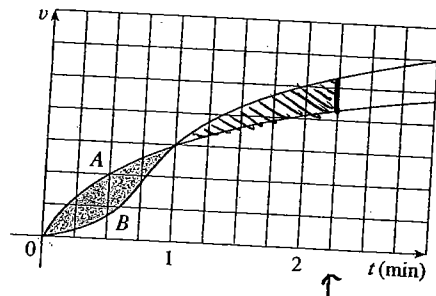
$$x = \frac{1}{2}(y+1)$$



$$\text{Area} = \int_0^1 \left(\frac{1}{2}(y+1) - \sqrt{y} \right) dy$$

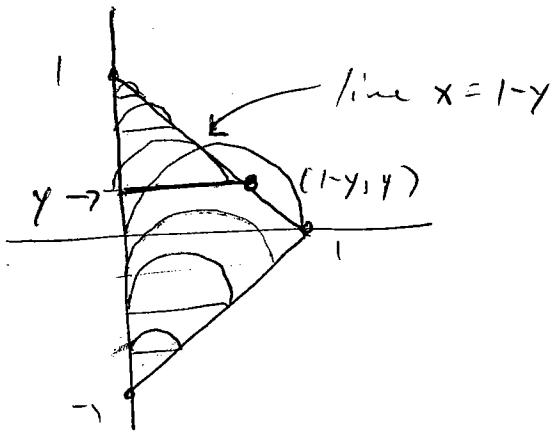
(b) Two cars, A and B, start side by side and accelerate from rest. The figure shows the graphs of their velocity functions.

- Which car is ahead after one minute?
- What is the meaning of the area of the shaded region?
- Estimate the time at which the two cars are again side by side.



- Car A.
- The distance between the cars after one minute
- $\approx 2\frac{1}{4}$ minutes
(when second shaded area equals first)

4. The base of a solid is a triangle with corners at $(0, 1)$, $(0, -1)$, and $(1, 0)$. Each cross-section perpendicular to the y -axis is a semicircle. Find the volume of the solid.



For $y \geq 0$, slice at $y =$
semicircle of diameter $1-y$:



$$\text{Area}(y) = \frac{1}{2} \pi \left(\frac{1-y}{2} \right)^2$$

By symmetry, volume = twice volume of the part $y \geq 0$.

$$\text{Vol} = 2 \int_0^1 \frac{1}{2} \pi \left(\frac{1-y}{2} \right)^2 dy$$

$$= \frac{\pi}{4} \int_0^1 (1 - 2y + y^2) dy$$

$$= \frac{\pi}{4} \left[y - y^2 + \frac{1}{3} y^3 \right]_0^1$$

$$= \boxed{\frac{\pi}{12}}$$

5(a) What is the average value of $f(x) = 2 + 6x - 3x^2$ on the interval $[0, b]$? For what values of b is this average equal to 4?

$$\begin{aligned} \text{Avg} &= \frac{1}{b} \int_0^b (2 + 6x - 3x^2) dx = \frac{1}{b} \left[2x + 3x^2 - x^3 \right]_0^b \\ &= \frac{1}{b} (2b + 3b^2 - b^3) \\ &= \boxed{2 + 3b - b^2} \end{aligned}$$

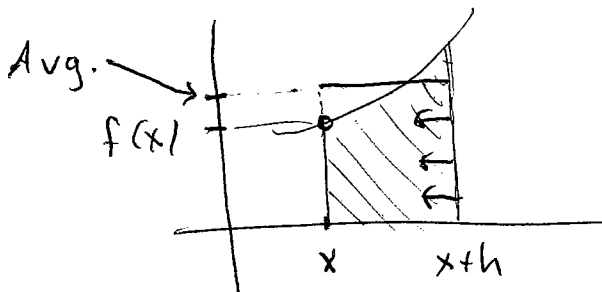
Avg is 4 when $4 = 2 + 3b - b^2$

$$b^2 - 3b + 2 = 0$$

$$(b-2)(b-1) = 0$$

$$\boxed{\text{when } b = 1, 2}$$

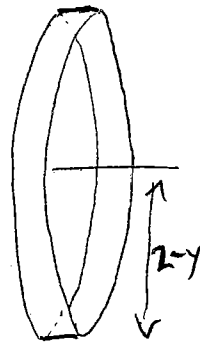
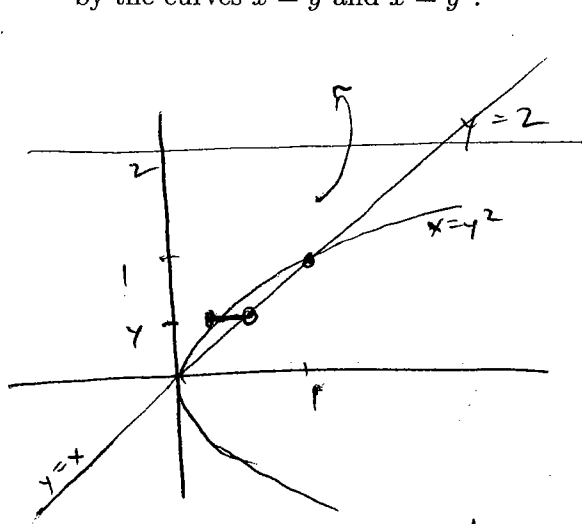
(b) If $f(x)$ is a continuous function, what is the limit as $h \rightarrow 0$ of the average value of f on the interval $[x, x+h]$? (A picture may help.)



As $h \rightarrow 0$, the average height of the shaded region approaches $f(x)$.

$$\text{i.e. } \lim_{h \rightarrow 0} \left(\text{avg. value of } f \text{ on } [x, x+h] \right) = \boxed{f(x)}$$

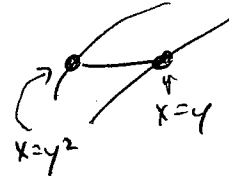
6(a) Find the volume of the solid obtained by rotating about the line $y = 2$ the region bounded by the curves $x = y$ and $x = y^2$.



Shells:

$$\text{perimeter} = 2\pi(2 - y)$$

$$\text{height} = y - y^2$$



$$\text{Area}(y) = 2\pi(2 - y)(y - y^2)$$

$$\text{Volume} = \int_0^1 2\pi(2 - y)(y - y^2) dy = 2\pi \int_0^1 (2y - 3y^2 + y^3) dy$$

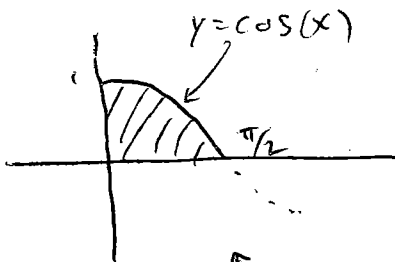
$$= 2\pi \left[y^2 - y^3 + \frac{1}{4}y^4 \right]_0^1$$

$$= \boxed{\frac{\pi}{2}}$$

Washers also work.

(b) Describe the solid whose volume is represented by the integral $\pi \int_0^{\pi/2} \cos^2(x) dx$.

$\uparrow \uparrow$
 $\pi r^2 = \text{Area of disk}$
 of radius $\cos(x)$;
 x ranges from 0 to $\pi/2$.



This region, rotated about the x -axis, has the given volume.