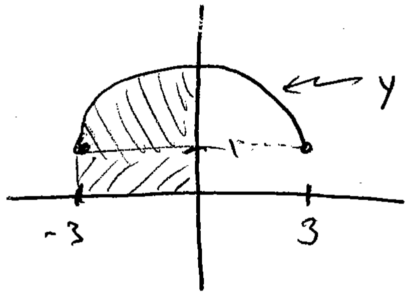
1(a) (5 points) Find the definite integral $\int_{-3}^{0} (1 + \sqrt{9 - x^2})\, dx$ by interpreting it as an area.

circle, radius 3



$y = 1 + \sqrt{9 - x^2}$

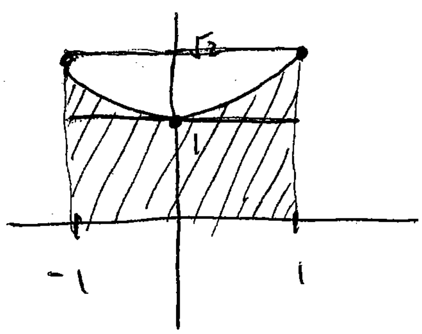
$$\text{Area} = \frac{1}{4}\pi(3)^2 + 3$$

$$= \boxed{\frac{9}{4}\pi + 3.}$$

(b) (5 points) What are the extreme values of $y = \sqrt{1 + x^2}$ on the interval $[-1, 1]$? Find numbers A and B such that $A \leq \int_{-1}^{1} \sqrt{1 + x^2}\, dx \leq B$, and explain your reasoning with a picture.

$y = \sqrt{1+x^2}$ is smallest when $x = 0$, largest when $x = \pm 1$

so $1 \leq \sqrt{1 + x^2} \leq \sqrt{2}$ on $[-1, 1]$



$$\boxed{2 \leq \int_{-1}^{1} \sqrt{1+x^2}\, dx \leq 2\sqrt{2}}$$

area of short rectangle

area of tall rectangle

The area represented by the integral contains the short rectangle and is contained in the large rectangle. Hence the inequalities hold.

2. (10 points)

(a) Find the derivatives of $\int_1^x \sqrt{1-s^2}\,ds$ and of $\int_{14}^{\sin x} \sqrt{1-s^2}\,ds$.

If $F(x) = \int_1^x \sqrt{1-s^2}\,ds$, $F'(x) = \boxed{\sqrt{1-x^2}}$.

$G(x) = \int_{14}^{x} \sqrt{1-s^2}\,ds$ also has derivative $\sqrt{1-x^2}$.

The derivative of $G(\sin x)$ is

$\boxed{\sqrt{1-\sin^2 x}\ \cos x}$ by the chain rule.

$= |\cos x| \cos x$.

UPDATE: the second integral isn't defined! The integrand has domain $[-1,1]$.

(b) Find an antiderivative $F(x)$ of $f(x) = \cos(x^2)$ with the property that $F(-10) = 0$.

$$\boxed{F(x) = \int_{-10}^{x} \cos(t^2)\,dt}$$

(c) Suppose you purchased a car seven years ago for \$15,000. Let $f(t)$ be the depreciation rate of the car at time t, where t is the time (in years) since you bought it. Write down an expression that represents the value of the car today.

$f(t)$ = dep. rate $\Rightarrow$ total depreciation is $\int_0^7 f(t)\,dt$.

Value today $= \boxed{15{,}000 - \int_0^7 f(t)\,dt}$

3. (10 points) Evaluate the integral, if it exists, or explain why it does not.

(a) $\int_0^3 |x^2 - 4|\, dx$

$x^2 - 4 \geq 0$ when $x \geq 2$

$x^2 - 4 \leq 0$ when $x \leq 2$

$$= \int_0^2 (4-x^2)dx + \int_2^3 (x^2-4)\,dx = \left(4x - \tfrac{1}{3}x^3\right)\Big|_0^2 + \left(\tfrac{1}{3}x^3 - 4x\right)\Big|_2^3$$

$$= \left(8 - \tfrac{8}{3}\right) + \left(9 - 12 - \tfrac{8}{3} + 8\right) = \boxed{13 - \tfrac{16}{3}}$$

(b) $\int_0^1 (4 - 3x)^{12}\, dx$

$u = 4 - 3x \quad du = -3dx \quad dx = \tfrac{-1}{3} du$

$$= \tfrac{-1}{3}\int_4^1 u^{12} du = \tfrac{-1}{3} \cdot \tfrac{1}{13} u^{13}\Big|_4^1$$

$$= \boxed{\frac{4^{13}}{39} - \frac{1}{39}}$$

(c) $\int_0^{13} \frac{dx}{\sqrt[3]{(1 + 2x)^2}}$

$u = 1 + 2x \quad du = 2dx \quad dx = \tfrac{1}{2} du$

$$= \tfrac{1}{2}\int_1^{27} u^{-2/3} du = \tfrac{1}{2} \cdot 3 u^{1/3}\Big|_1^{27}$$

$$= \tfrac{3}{2}\left(27^{1/3} - 1^{1/3}\right) = \boxed{3}.$$

(d) $\int \cos(\pi t) \sin^3(\pi t)\, dt$

$u = \sin(\pi t) \quad du = \pi \cos(\pi t) dt$

$dt = \frac{1}{\pi \cos(\pi t)} dt$

$$= \int \tfrac{1}{\pi} u^3 du = \tfrac{1}{4\pi} u^4 + C$$

$$= \boxed{\frac{1}{4\pi} \sin^4(\pi t) + C}$$

4. (10 points)

(a) Write down both parts of the Fundamental Theorem of Calculus, including all the requirements.

If f is continuous on $[a,b]$ then

I: the function $g(x) = \int_a^x f(t)\,dt$ $(a \le x \le b)$ is continuous and has derivative $f(x)$

II: $\int_a^b f(x)\,dx = F(b) - F(a)$ where F is any antiderivative of f.

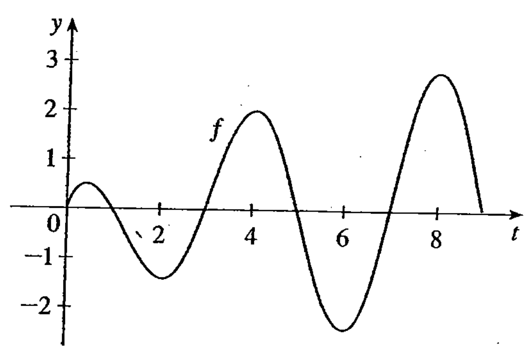
(b) Let $f(t)$ be the function whose graph is given below and let $g(x) = \int_0^x f(t)\,dt$. Find the intervals or points at which:

$\boxed{g'(x) = f(x)}$

(i) $g(x)$ has a local maximum: $\boxed{x = 1, 5, 9}$

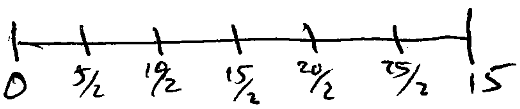
(ii) $g(x)$ is concave down: $\boxed{\tfrac{1}{2} \le x \le 2,\quad 4 \le x \le 6,\quad 8 \le x \le 9}$ (i.e. $f'(x) < 0$)

(i) $g(x)$ is increasing: $\boxed{0 \le x \le 1,\quad 3 \le x \le 5,\quad 7 \le x \le 9}$



5(a) (10 points) Use six rectangles to estimate the area under the curve $y = e^{-x^2}$ between $x = 0$ and $x = 15$ as follows.

(a) Write down the sample points (choose left or right, and say which).



Right endpoints:

$$\boxed{\begin{array}{ll} x_1^* = 5/2 & x_4^* = 20/2 \\ x_2^* = 10/2 & x_5^* = 25/2 \\ x_3^* = 15/2 & x_6^* = 30/2 \end{array}}$$

(b) Write out a complete expression for the estimate (you do not need to evaluate the number).

Total area of six rectangles: $\Delta x = 15/6 = 5/2$

$$\boxed{\text{Area} = \frac{5}{2}\left(e^{-(5/2)^2} + e^{-(10/2)^2} + e^{-(15/2)^2} + e^{-(20/2)^2} + e^{-(25/2)^2} + e^{-(30/2)^2} \right)}$$

(c) Express part (b) using sigma notation. $x_i^* = 5i/2 \quad \left(= \frac{15i}{6}\right)$

$$\boxed{\text{Area} = \sum_{i=1}^{6} \frac{5}{2} e^{-\left(\frac{5i}{2}\right)^2}}$$

(d) Write down an expression for the area estimate given by using n rectangles (use sigma notation if you like).

$$x_i^* = \frac{15i}{n}, \quad \Delta x = \frac{15}{n}$$

$$\boxed{\text{Area} = \sum_{i=1}^{n} \frac{15}{n} e^{-\left(\frac{15i}{n}\right)^2}}$$

6 (10 points) When a raindrop falls, its size increases and so its air resistance increases also. Suppose a raindrop has an initial downward velocity of 10 m/s and its downward accceleration is given by $a(t) = 9 - 0.9t$ for time $0 \le t \le 10$ and $a(t) = 0$ for $t \ge 10$.

(a) What is the downward velocity at time $t = 10$?

$v(t) = 9t - \frac{9}{10}\cdot\frac{1}{2}t^2 + C \qquad v(0) = 10$

$v(0) = 0 - 0 + C = 10, \quad C = 10.$

$v(t) = 9t - \frac{9}{20}t^2 + 10.$

$v(10) = 90 - \frac{9}{20}(100) + 10 = 90 - 45 + 10 = \boxed{55 \text{ m/s}}$

(b) What is the downward velocity at time $t = 15$? $a(t) = 0 \Rightarrow v(t)$ is constant.

$\Rightarrow v(t) =$ vel. at time 10

$= \boxed{55 \text{ m/s.}}$

(c) How far has the raindrop fallen at time $t = 10$ and at time $t = 15$?

Time t=10

$\text{dist}(t) = \int_0^t v(s)\,ds = \frac{9}{2}t^2 - \frac{3}{20}t^3 + 10t + D$

initial distance fallen $= 0$ so $D = 0$.

$\text{dist}(10) = \frac{9}{2}(10)^2 - \frac{3}{20}(10)^3 + 10(10)$

$= 450 - 150 + 100 = \boxed{400 \text{ m}}$

At t=15: add 55 m per second past 10

$\Rightarrow$ add 55×5 m.

$\text{dist} = 400 + 275 = \boxed{675 \text{ m}}$