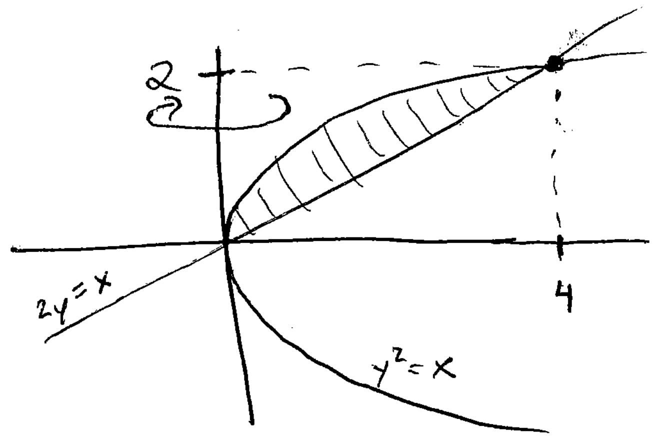
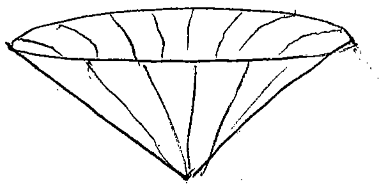
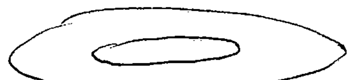
**1.** (10 points) Find the volume of the solid obtained by rotating the region between the curves $y^2 = x$ and $x = 2y$ about the $y$-axis. Sketch the region, the solid, and the typical disk, washer, or shell.

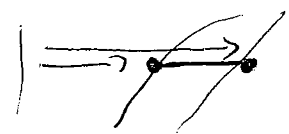
$y^2 = 2y \Rightarrow y^2 - 2y = 0 \Rightarrow y = 0,\ y = 2$
$\Downarrow$
$x = 4$.



Washers:





At $y$, inner radius $= y^2$
outer radius $= 2y$

$$\text{Vol} = \int_0^2 \pi(2y)^2 - \pi(y^2)^2\,dy$$

$$= \pi\int_0^2 4y^2 - y^4\,dy = \pi\left[\frac{4}{3}y^3 - \frac{1}{5}y^5\right]_0^2$$

$$= \pi\left(\frac{32}{3} - \frac{32}{5}\right) = \boxed{\frac{64\pi}{15}}$$

Shells: At $x$, height is
$\sqrt{x} - \frac{1}{2}x$

$$\text{Vol} = \int_0^4 2\pi x\left(\sqrt{x} - \tfrac{1}{2}x\right)dx = \pi\int_0^4 2x^{3/2} - x^2\,dx$$

$$= \pi\left[\frac{4}{5}x^{5/2} - \frac{1}{3}x^3\right]_0^4 = \pi\left(\frac{4}{5}(32) - \frac{1}{3}(64)\right)$$

$$= \boxed{\frac{64\pi}{15}}.$$

**2(a)** (6 points) Make a substitution to express the integrand as a rational function, and evaluate the integral: $\int \frac{\sqrt{x}}{x^2+x}\,dx$.

$u = \sqrt{x} \rightarrow u^2 = x \qquad 2u\,du = dx$

$$\int \frac{u}{u^4+u^2}\,2u\,du = \int \frac{2}{u^2+1}\,du$$

$$= 2\tan^{-1}(u) + C$$

$$= \boxed{2\tan^{-1}(\sqrt{x}) + C}$$

**2(b)** (6 points) Use a trigonometric substitution to evaluate $\int \frac{x^2}{(4-x^2)^{3/2}}\,dx$.

$\hookrightarrow x = 2\sin\theta$

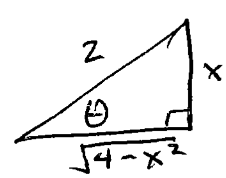
$dx = 2\cos\theta\,d\theta$

$$\int \frac{(2\sin\theta)^2}{(4-4\sin^2\theta)^{3/2}}(2\cos\theta)\,d\theta$$

$$= \int \frac{(2\sin\theta)^2}{(2\cos\theta)^3}(2\cos\theta)\,d\theta$$

$$= \int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$$

$$= \tan\theta - \theta + C$$




$$= \tan\left(\sin^{-1}\left(\tfrac{x}{2}\right)\right) - \sin^{-1}\left(\tfrac{x}{2}\right) + C$$

$$= \boxed{\frac{x}{\sqrt{4-x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + C}$$

**3.** (18 points) Evaluate the integrals:

(a) $\int \sin^5 x\, dx$

$= \frac{-1}{5}\sin^4 x\cos x + \frac{4}{5}\int \sin^3 x\, dx$

$= \frac{-1}{5}\sin^4 x\cos x + \frac{4}{5}\left(-\frac{1}{3}\sin^2 x\cos x + \frac{2}{3}\int \sin x\, dx\right)$

$= \boxed{\frac{-1}{5}\sin^4 x\cos x - \frac{4}{15}\sin^2 x\cos x + \frac{-8}{15}\cos x + C}$

(b) $\int \tan^3 x\sec^2 x\, dx$ $= \int (\sec^2 x - 1)\sec x\,(\sec x\tan x)\, dx$

$u = \sec x$
$du = \sec x\tan x\, dx$

$= \int (u^2-1)u\, du$

$= \int u^3 - u\, du = \frac{1}{4}u^4 - \frac{1}{2}u^2 + C$

$= \boxed{\frac{1}{4}\sec^4 x - \frac{1}{2}\sec^2 x + C}$

OR:

$u = \tan x$
$du = \sec^2 x\, dx$

$\int u^3 du = \frac{1}{4}u^4 + C$

$= \boxed{\frac{1}{4}\tan^4 x + C}$

(these families give the same functions)

(c) $\int 3^{\sin t} \cos t \, dt$

$u = \sin t$
$du = \cos t \, dt$

$$= \int 3^u du = \frac{1}{\ln(3)} 3^u + C$$

$$= \boxed{\frac{1}{\ln(3)} 3^{\sin t} + C}$$

**4.** (6 points) Does $\displaystyle\int_1^\infty \frac{\cos^2 x}{1+x^2}\,dx$ converge or diverge? Use a comparison to explain why.

We have $0 \leq \dfrac{\cos^2 x}{1+x^2} \leq \dfrac{1}{1+x^2} \leq \dfrac{1}{x^2}$

(since $\cos^2 x \leq 1$)

and $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges $( = 1 )$

so by the comparison theorem,

$$\boxed{\int_1^\infty \frac{\cos^2 x}{1+x^2}\,dx \text{ converges.}}$$

**5(a)** (4 points) Find $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$.

Indeterminate of type $\frac{0}{0}$

$$\underset{\text{L'Hospital}}{=} \lim_{x \to 0} \frac{\sin x}{2x}$$

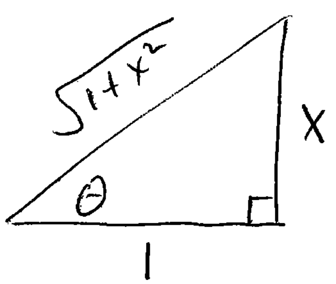
Indeterminate of type $\frac{0}{0}$

$$\underset{\text{L'Hospital}}{=} \lim_{x \to 0} \frac{\cos x}{2} = \boxed{\frac{1}{2}}.$$

**(b)** (4 points) Simplify the expression $\cos(\tan^{-1} x)$.

Let $\theta = \tan^{-1} x$

so $\tan \theta = x$




$$\cos \theta = \boxed{\frac{1}{\sqrt{1+x^2}}}$$

**6(a)** (6 points) Write out the form of the partial fraction decomposition. Do not determine the numerical values of the coefficients. [Hint: be careful.]

**(i)** $\dfrac{x+1}{(x^2+4)(x-2)^2}$

$$\boxed{\frac{Ax+B}{x^2+4} + \frac{C}{x-2} + \frac{D}{(x-2)^2}}$$

**(ii)** $\dfrac{18}{x^4+2x^3+3x^2} = \dfrac{18}{x^2(x^2+2x+3)}$

does not factor

$$\boxed{\frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+2x+3}}$$

**(iii)** $\dfrac{x^2+1}{(x+1)(x^2-1)(x+2)}$

$$= \frac{x^2+1}{(x+1)(x+1)(x-1)(x+2)}$$

$$\boxed{\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1} + \frac{D}{x+2}}$$

**6(b)** (6 points) Evaluate $\displaystyle\int_0^2 \frac{x^3}{x-3}\,dx$.

$$\begin{array}{r}
x^2+3x+9 \quad + \dfrac{27}{x-3} \\
x-3 \,\big)\, x^3+0x^2+0x+0 \\
x^3-3x^2 \qquad\qquad \\
\hline
3x^2+0x \quad \\
3x^2-9x \quad \\
\hline
9x+0 \\
9x-27 \\
\hline
27
\end{array}$$

$$\int_0^2 \left(x^2+3x+9+\frac{27}{x-3}\right)dx$$

$$= \frac{1}{3}x^3 + \frac{3}{2}x^2 + 9x + 27\ln|x-3| \Big|_0^2$$

$$= \frac{8}{3} + \frac{12}{2} + 18 + 27\ln|-1|$$

$$-0-0-0-27\ln|-3|$$

$$= \boxed{\frac{8}{3}+6+18-27\ln(3)}$$

**7(a)** (6 points) Use integration by parts twice to evaluate $\int e^x \cos(x)\,dx$.

$u = e^x \quad dv = \cos x\,dx$

$du = e^x dx \quad v = \sin x$

$$= e^x \sin x - \int e^x \sin x\,dx$$

$u = e^x \quad dv = \sin x\,dx$

$du = e^x dx \quad v = -\cos x$

$$= e^x \sin x - \left[ -e^x \cos x - \int -e^x \cos x\,dx \right]$$

So $$\int e^x \cos x\,dx = e^x \sin x + e^x \cos x + \int e^x \cos x\,dx$$

$$2\int e^x \cos x\,dx = e^x(\sin x + \cos x)$$

$$\int e^x \cos x\,dx = \boxed{\frac{e^x}{2}(\sin x + \cos x) + C}$$

**(b)** (3 points) Write down an antiderivative of $f(x) = e^{\cos^3 x}$ which is zero when $x = 2$.

$$\boxed{F(x) = \int_2^x e^{\cos^3 t}\,dt}$$

**(c)** (3 points) What is the average value of $f(x) = 12x^3 - 3x^2 + 12$ on the interval $[-2, 2]$?

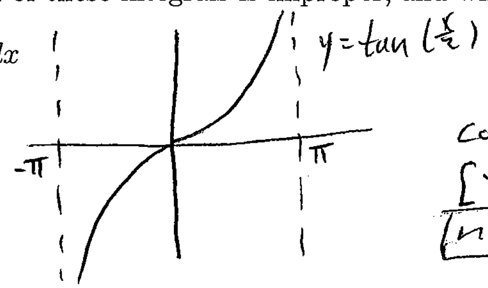
$$\frac{1}{2-(-2)} \int_{-2}^{2} (12x^3 - 3x^2 + 12)\,dx = \frac{1}{4}\left(3x^4 - x^3 + 12x\right)\Big|_{-2}^{2}$$

$$= \frac{1}{4}\left(3(2)^4 - 8 + 24 - 3(-2)^4 + (-8) - 12(-2)\right)$$

$$= \frac{1}{4}(32) = \boxed{8}.$$

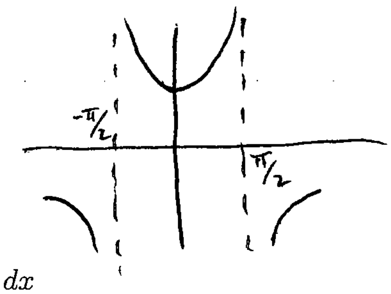
**8.** (6 points) Which of these integrals is improper, and why?

**(a)** $\int_{-\pi/2}^{\pi/2} \tan(x/2)\, dx$




The integrand is continuous on the interval $[-\pi/2, \pi/2]$ so it's $\boxed{\text{not improper.}}$

**(b)** $\int_{0}^{\pi} \sec(x)\, dx$




$\sec(x)$ is undefined at $x = \pi/2$, so $\boxed{\text{improper}}$.

**(c)** $\int_{0}^{2} \frac{x+1}{x^2+3x-10}\, dx$

$\int_0^2 \frac{x+1}{(x+5)(x-2)}\, dx$

Integrand is undefined at $x = 2$ so $\boxed{\text{improper}}$.

**9.** (6 points)
**(a)** If $w'(t)$ is the rate of growth of a child in pounds per year, what does $\int_5^{10} w'(t)\, dt$ represent?

The child's weight gain (in pounds) between year 5 and year 10.

**(b)** Suppose a particle moves back and forth along a straight line with velocity $v(t)$, measured in feet per second. What is the meaning of $\int_{60}^{120} |v(t)|\, dt$?

total distance (in feet) travelled (in either direction) during the second minute.