1(a) A honeybee population starts at 100 and increases at a rate of $n'(t)$ bees per week. What does $100 + \int_0^{18} n'(t)\,dt$ represent?

The total population after 18 weeks

(b) Give a number x such that $\sin^{-1}(\sin(x)) = x$.

any x in $[-\pi/2, \pi/2]$

(c) Give a number y such that $\sin^{-1}(\sin(y)) \neq y$.

any y not in $[-\pi/2, \pi/2]$

(d) Find the average value of $\frac{(\ln x)^2}{x}$ over the interval $[1, e^3]$.

$$\text{avg} = \frac{1}{e^3-1} \int_1^{e^3} \frac{(\ln x)^2}{x}\,dx \qquad u = \ln x, \quad du = \frac{1}{x}\,dx$$

$$= \frac{1}{e^3-1} \int_0^3 u^2\,du = \frac{1}{e^3-1}\left[\frac{1}{3}u^3\right]_0^3$$

$$= \frac{1}{e^3-1}\left[\frac{1}{3}\cdot 27\right] = \boxed{\frac{9}{e^3-1}}$$

2(a) Does $\int_0^\infty \frac{x}{x^3+1}\,dx$ converge or diverge? Use a comparison to explain why.

For $x \geq 1$, $\frac{x}{x^3+1} \leq \frac{x}{x^3} = \frac{1}{x^2}$

and $\int_1^\infty \frac{1}{x^2}\,dx$ converges

$\Longrightarrow \int_1^\infty \frac{x}{x^3+1}\,dx$ converges.

Also, $\int_0^1 \frac{x}{x^3+1}\,dx$ is a finite number.

So $\boxed{\int_0^\infty \frac{x}{x^3+1}\,dx \text{ converges}}$.

(b) Explain why $\int_3^5 \frac{1}{\sqrt{5-x}}\,dx$ is improper, and evaluate it.

Improper because $\frac{1}{\sqrt{5-x}}$ is undefined at 5.

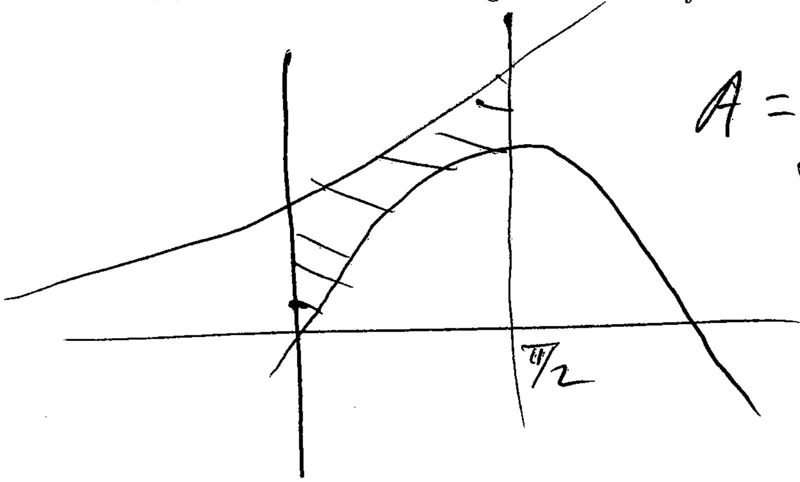
$$\lim_{b\to 5^-} \underbrace{\int_3^b \frac{1}{\sqrt{5-x}}\,dx}_{\parallel}$$

$u = 5-x$
$du = -dx$

$$\int_2^{5-b} -u^{-1/2}\,du = -2u^{\frac{1}{2}}\Big|_2^{5-b}$$

$$\lim_{b\to 5^-}\left[-2(5-b)^{1/2} + 2(2)^{1/2}\right] = \boxed{2\sqrt{2}}$$

3(a) Find the area of the region bounded by the curves $y = e^x$, $y = \sin x$, $x = 0$, and $x = \pi/2$.



$$A = \int_0^{\pi/2} e^x - \sin x \, dx$$

$$= e^x + \cos x \Big|_0^{\pi/2}$$

$$= e^{\pi/2} - e^0 + \cos(\pi/2) - \cos(0)$$

$$= \boxed{e^{\pi/2} - 2}$$

(b) Evaluate $\int_0^2 y^2 \sqrt{1+y^3} \, dy$.

$u = 1 + y^3$
$du = 3y^2 dy$

$$\frac{1}{3}\int_1^9 \sqrt{u} \, du$$

$$= \frac{1}{3} \cdot \frac{2}{3} u^{3/2} \Big|_1^9$$

$$= \frac{2}{9}(27 - 1)$$

$$= \boxed{\frac{52}{9}}$$

4(a) Use a trigonometric substitution to find $\int \frac{1}{x^2\sqrt{x^2+4}}\,dx$.

$x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$$\int \frac{1}{4\tan^2\theta\, 2\sec\theta}\, 2\sec^2\theta\, d\theta = \int \frac{1}{4}\,\frac{\cos\theta}{\sin^2\theta}\, d\theta$$

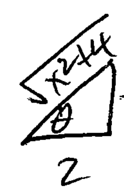
$u = \sin\theta, \quad du = \cos\theta\, d\theta$

$$\frac{1}{4}\int \frac{1}{u^2}\,du = \frac{-1}{4u} + C$$

$$= \frac{-1}{4\sin\theta} + C$$

$$= \boxed{\frac{-1}{4\sin(\tan^{-1}(\frac{x}{2}))} + C}$$

$$\left(= \frac{-1}{4x}\sqrt{x^2+4} + C \quad \text{using} \right)$$



(b) Find $\lim_{x\to 0} \frac{x}{\tan^{-1}(4x)}$.

Indeterminate, type $\frac{0}{0}$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{1}{\frac{4}{1+(4x)^2}} = \boxed{\frac{1}{4}}$$

5(a) Write out the appropriate form of the partial fraction decomposition. Do *not* determine the numerical values of the coefficients. [Hint: be careful.]

(i) $\frac{x+5}{(x^2+4)(x-2)^2}$ $= \boxed{\frac{Ax+B}{x^2+4} + \frac{C}{x-2} + \frac{D}{(x-2)^2}}$

(ii) $\frac{x^2+1}{(x+1)(x^2-1)(x+2)}$ $= \frac{x^2+1}{(x+1)^2(x-1)(x+2)}$

$$= \boxed{\frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{x-1} + \frac{D}{x+2}}$$

(b) Evaluate $\int_0^2 \frac{x^3}{x-2}\,dx$.

$$\begin{array}{r} x^2 + 2x + 4 \quad + \frac{8}{x-2} \\ x-2 \,\big)\, x^3 + 0x^2 + 0x + 0 \\ x^3 - 2x^2 \\ \hline 2x^2 + 0 \\ 2x^2 - 4x \\ \hline 4x + 0 \\ 4x - 8 \\ \hline 8 \end{array}$$

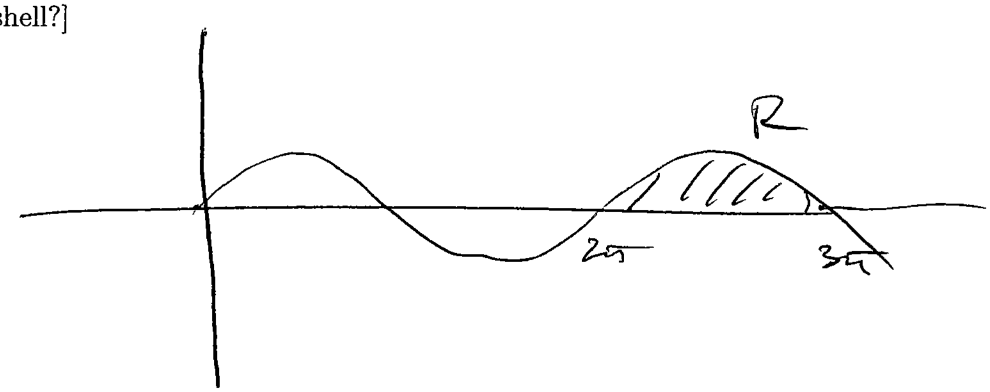
$$= \int_0^2 x^2+2x+4\,dx + \int_0^2 \frac{8}{x-2}\,dx$$

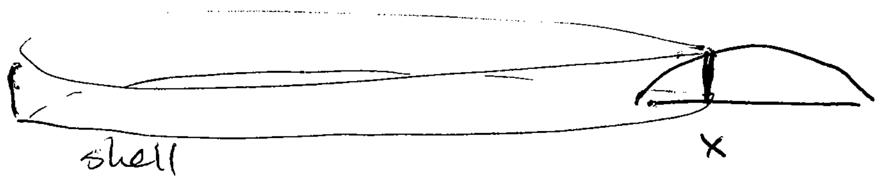
improper at $x=2$

$$= \frac{1}{3}x^3 + x^2 + 4x\Big|_0^2 + \lim_{b\to 2^-}\Big(\underbrace{8\ln|b-2|}_{\to -\infty} - 8\ln|-2|\Big)$$

$\boxed{\text{diverges to } -\infty}$

6. Let R be the region bounded by $y = \sin x$, $x = 2\pi$, $x = 3\pi$, and $y = 0$. Use shells to find the volume of the solid obtained by rotating R about the y-axis. [Draw pictures. What is the area of a typical shell?]





radius $= x$ height $= \sin x$

$A(x) = 2\pi x \sin x$

$$\text{Volume} = \int_{2\pi}^{3\pi} 2\pi x \sin x \, dx$$

Parts

$u = 2\pi x$ $dv = \sin x \, dx$

$du = 2\pi \, dx$ $v = -\cos x$

$$= -2\pi x \cos x \Big|_{2\pi}^{3\pi} - \int_{2\pi}^{3\pi} -2\pi \cos x \, dx$$

$$= -2\pi(3\pi)(-1) + 2\pi(2\pi)(1) + 2\pi\left(\sin(3\pi) - \sin(2\pi)\right)$$

0

$$= \boxed{10\pi^2}$$

$\sin^2 x + \cos^2 x = 1$
$1 + \tan^2 x = \sec^2 x$
$\sin^2 x = \frac{1}{2}(1 - \cos(2x))$
$\cos^2 x = \frac{1}{2}(1 + \cos(2x))$
$\sin(2x) = 2 \sin x \cos x$
$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$
$\sin A \cos B = \frac{1}{2}(\sin(A - B) + \sin(A + B))$
$\sin A \sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B))$
$\cos A \cos B = \frac{1}{2}(\cos(A - B) + \cos(A + B))$

Extra Credit Use logarithmic differentiation to find $\frac{dy}{dx}$ when $y = x^{(x^x)}$.

$$\ln y = \ln\left(x^{(x^x)}\right) = x^x \ln x$$

$$\frac{d}{dx}(\ln y) = x^x \frac{d}{dx}(\ln x) + \frac{d}{dx}(x^x)\ln x$$

$$y = x^x$$
$$\ln y = x \ln x$$
$$\frac{1}{y}\frac{dy}{dx} = x\frac{1}{x} + \ln x$$
$$\boxed{\frac{dy}{dx} = x^x(1+\ln x)}$$

So

$$\frac{1}{y}\frac{dy}{dx} = x^x\frac{1}{x} + x^x(1+\ln x)\ln x$$

$$\frac{dy}{dx} = x^{(x^x)}\left[x^x\frac{1}{x} + x^x(1+\ln x)\ln x\right]$$