Solution to some Problems of Homework 6

Section 2.8:

8 - Suppose $4x^2+9y^2=36$, where x and y are functions of t.

a. If $\frac{dy}{dt}=\frac{1}{3}$, find $\frac{dx}{dt}$ when $x=2$ and $y=\frac{2}{3}\sqrt{5}$.

b. If $\frac{dx}{dt}=3$, find $\frac{dy}{dt}$ when $x=-2$ and $y=\frac{2}{3}\sqrt{5}$.

Suppose $4x^2+9y^2=36$ where x and y are function of t.

Differentiating w.r.t. to t and using the chain rule:

$$8x\frac{dx}{dt}+18y\frac{dy}{dt}=0$$

a. $\frac{dy}{dt}=\frac{1}{3}$, $\frac{dx}{dt}=?$ when $x=2$ & $y=\frac{2}{3}\sqrt{5}$

So. $16\frac{dx}{dt}+12\sqrt{5}\cdot\frac{1}{3}=0$

$$16\frac{dx}{dt}=-4\sqrt{5}$$

and $\boxed{\frac{dx}{dt}=\frac{-4\sqrt{5}}{16}=-\frac{\sqrt{5}}{4}}$

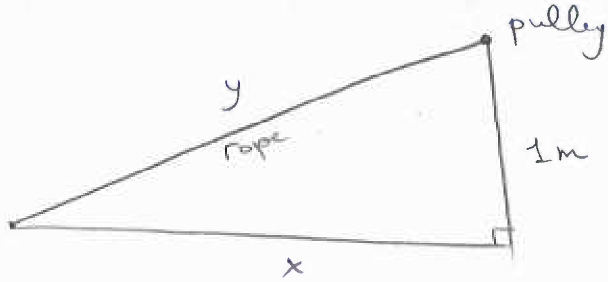
b. $\frac{dx}{dt}=3$ $\frac{dy}{dt}=?$ when $x=-2$ and $y=\frac{2}{3}\sqrt{5}$

$$(-16)(3)+12\sqrt{5}\cdot\frac{dy}{dt}=0$$

$$12\sqrt{5}\frac{dy}{dt}=48$$

$$\boxed{\frac{dy}{dt}=\frac{48}{12\sqrt{5}}=\frac{4}{\sqrt{5}}}$$

20. A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1m higher than the bow of the boat.
If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the dock?



Let x be the horizontal distance travelled by the boat.
Let y be the distance from the boat to the pulley.

Given: $\frac{dy}{dt} = -1\,m/s$

Unknown: $\frac{dx}{dt} = ?$ when $x = 8m$.

Relation: $x^2 + 1 = y^2$

Differentiate both sides with respect to t:

$$2x\frac{dx}{dt} = 2y\frac{dy}{dt}$$

So, $$x\frac{dx}{dt} = y\frac{dy}{dt}$$

$$\frac{dx}{dt} = \frac{y}{x}\cdot\underbrace{\frac{dy}{dt}}_{\frac{dy}{dt}=-1} = (-1)\cdot\frac{y}{x} = \frac{-y}{x}$$

Now, when $x = 8m$,

$$y = \sqrt{64+1} = \sqrt{65}$$

So, $$\boxed{\frac{dx}{dt} = \frac{-\sqrt{65}}{8}\ m/s}$$

Section 2.9:

4- Find the linearization $L(x)$ of the function at a.

$f(x) = x^{3/4}$, $a = 16$.

The linearization of $f(x)$ at a is given by

$L(x) = f'(a)(x-a) + f(a)$

$f(x) = x^{3/4}$, $f(a) = (16)^{3/4} = (2^4)^{3/4} = 2^3 = 8$

$f'(x) = \frac{3}{4} \cdot x^{-1/4}$, $f'(a) = \frac{3}{4} \cdot (16)^{-1/4} = \frac{3}{4} \cdot (2^4)^{-1/4} = \frac{3}{8}$

$L(x) = \frac{3}{8}(x - 16) + 8$

$= \frac{3}{8} x - 6 + 8 = \frac{3}{8} x + 2$

So, $\boxed{L(x) = \frac{3}{8} x + 2}$

16- a- Find the differential dy.
b- Evaluate dy for the given values of x and dx.

$y = \cos \pi x$, $x = \frac{1}{3}$, $dx = -0.02$

The differential dy of $f(x)$ is

$dy = f'(x) \cdot dx$.

a- $f(x) = \cos \pi x$

$f'(x) = -\sin(\pi x) \cdot \pi = -\pi \sin(\pi x)$.

So, $dy = -\pi \sin(\pi x) \cdot dx$

b- For $x = \frac{1}{3}$ and $dx = -0.02$,

$dy = -\pi \sin\left(\frac{\pi}{3}\right) \cdot (-0.02)$

$= 0.02\pi \cdot \frac{\sqrt{3}}{2} = (0.01) \sqrt{3}\, \pi$

$\boxed{dy = 0.01 \pi \sqrt{3} \approx 0.054}$

24. Use a linear approximation (or differentials) to estimate the given number.

$\sin 1^\circ$

Linear Approximation: $f(x) \approx f'(a)(x-a) + f(a)$

$f(x) = \sin x \qquad a = 0$

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

So, $L(x) = f'(0)(x-0) + f(0) = x$

Hence, $f(x) \approx x$

So for $x = 1^\circ = \frac{\pi}{180}$ radians, (Always change your angles to radians)

$\sin 1^\circ \approx \frac{\pi}{180} \approx 0.01745$

Differentials: $f(a+dx) \approx f(a) + dy$

$f(x) = \sin x$

$f'(x) = \cos x$

So, $dy = \cos x \, dx$

When $x = 0$, $dx = 1^\circ = \frac{\pi}{180}$

So, $dy = \cos 0 . dx = 1 . \frac{\pi}{180}$

and $dy = \frac{\pi}{180}$

But $\sin 1^\circ = f(1^\circ) \approx f(0) + dy = 0 + dy$

and $\sin 1^\circ \approx \frac{\pi}{180} \approx 0.01745$