Solution to Some Problems of Hw 15.

Section 5.4:

10- If the work required to stretch a spring 1 ft beyond its natural length is 12 ft-lb, how much work is needed to stretch it 9 inches beyond its natural length?

Step 1: Find the stiffness constant k of the spring.

$$12 = \int_0^1 kx\,dx = \frac{1}{2}kx^2\Big]_0^1 = \frac{1}{2}k$$

and $k = 24$ lb/ft.

Step 2: Find the work needed to stretch the spring 9 inches:

9 inch $= \frac{3}{4}$ ft

$$W = \int_0^{3/4} 24x\,dx$$

$$= 24\cdot\frac{x^2}{2}\Big]_0^{3/4} = 12\,x^2\Big]_0^{3/4} = \frac{27}{4} = 6.75 \text{ ft-lb.}$$

20- A circular swimming pool has a diameter of 24 ft, the sides are 5 ft high, and the depth of the water is 4 ft. How much work is required to pump all of the water out over the side? (Use the fact that water weighs 62.5 lb/ft³)

A horizontal cylindrical slice of water Δx ft thick has volume of $\pi r^2 h = \pi\cdot 12^2\cdot\Delta x$ ft³ and weighs about

$$(62.5 \text{ lb/ft}^3)\cdot(144\pi\,\Delta x \text{ ft}^3) = 9000\pi\,\Delta x \text{ lb.}$$

If the slice lies x_i^* ft below the edge of the pool (where $1 \le x_i^* \le 5$), then the work needed to pump it out is about

$$\text{weight}\cdot x_i^* = 9000\pi\,\Delta x\cdot x_i^*$$

Thus,

$$W = \lim_{n\to\infty}\sum_{i=1}^{n} 9000\pi\,x_i^*\,\Delta x = \int_1^5 9000\pi\,x\,dx$$

$$= 4500\pi\,x^2\Big]_1^5 = 4500\pi(25-1) = 108{,}000\,\pi \text{ ft-lb.}$$

Section 5.5:

2. Find the average value of the function on the given interval.

$f(x) = \sin 4x \quad , \quad [-\pi, \pi]$

$$f_{ave.} = \frac{1}{b-a} \cdot \int_a^b f(x)\,dx = \frac{1}{\pi - (-\pi)} \cdot \int_{-\pi}^{\pi} \sin(4x)\,dx = 0$$

Sine function is odd

<u>Note</u>: If f is an odd function i.e.

$$f(-x) = -f(x)$$

then $\int_{-a}^{a} f(x)\,dx = 0$ for any $a \in \mathbb{R}$.

10. a. Find the average value of f on the given interval.

b. Find c such that $f_{ave.} = f(c)$.

c. Sketch the graph of f and a rectangle whose area is the same as the area under the graph of f.

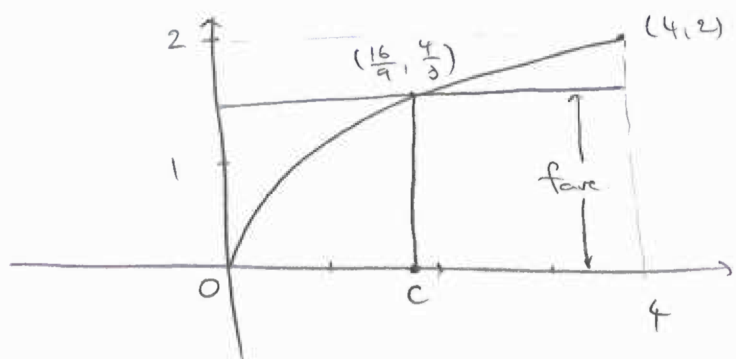
$f(x) = \sqrt{x} \quad , \quad [0, 4]$

a. $f_{ave} = \frac{1}{4-0} \cdot \int_0^4 \sqrt{x}\,dx = \frac{1}{4} \cdot \frac{2}{3} x^{3/2} \Big]_0^4 = \frac{1}{6} \, x^{3/2} \Big]_0^4$

$= \frac{1}{6} \cdot 8 = \frac{4}{3}$

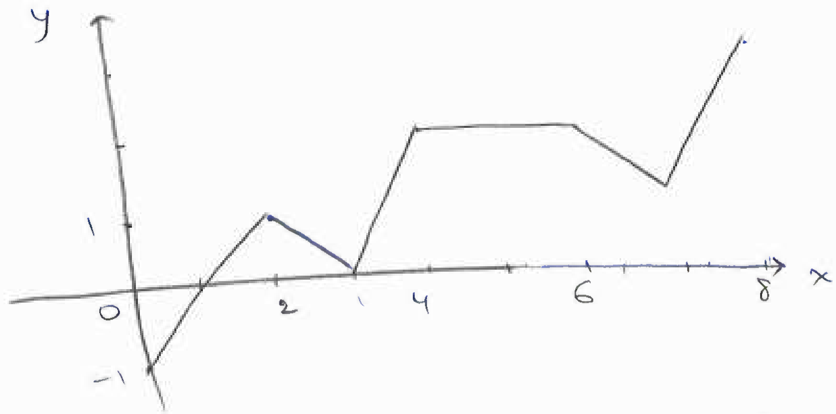
b. $f(c) = f_{ave.}$, so $\sqrt{c} = \frac{4}{3}$, $c = \frac{16}{9}$.

c.



<u>Note</u>: Rectangle with base $[0, 4]$ and height $f(c) = f_{ave} = \frac{4}{3}$ has the same area as the region under the graph of f from 0 to 4.

15 - find the average value of f on $[0, 8]$



Use geometric interpretations to find the values of the integrals.

$$\int_0^8 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx$$
$$+ \int_3^4 f(x)\,dx + \int_4^6 f(x)\,dx + \int_6^7 f(x)\,dx$$
$$+ \int_7^8 f(x)\,dx$$

$$= -\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + 1 + 4 + \frac{3}{2} + 2 = 9$$

Thus, the average value of f on $[0, 8]$ is

$$f_{ave} = \frac{1}{8-0} \cdot \int_0^8 f(x)\,dx = \frac{1}{8} \cdot 9 = \frac{9}{8}$$