

1(a) Draw the graph of a function $f(x)$ satisfying:

$$|f(x)| < 1 \text{ for } x > 2$$

$\lim_{x \rightarrow 2^+} f(x)$ does not exist

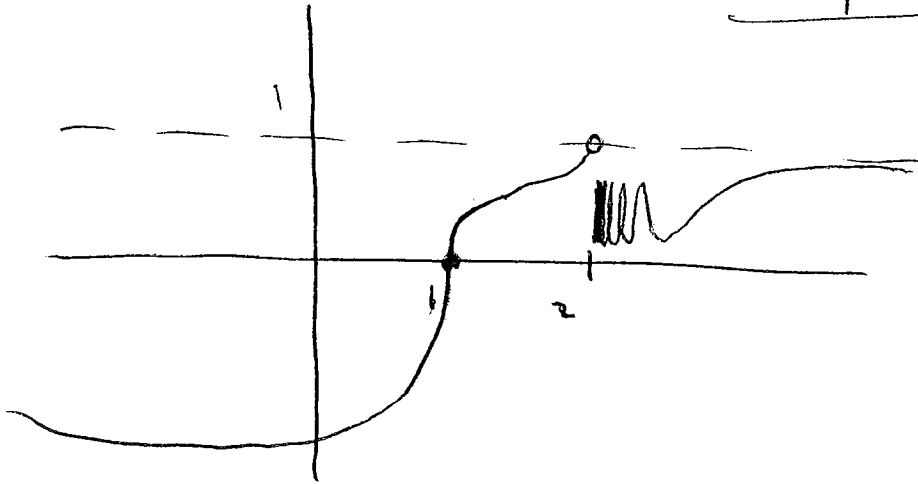
$$\lim_{x \rightarrow 2^-} f(x) = 1$$

$$\lim_{x \rightarrow \infty} f(x) = 1$$

$y = f(x)$ has a tangent line at every $x < 2$

$f(x)$ is continuous but not differentiable at $x = 1$

one possibility:



(b) Using a theorem from the course, show that $\lim_{x \rightarrow 0} x \sin(1/x) = 0$.

Note that $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$ for all x

$$\text{So } \underline{-x \leq x \sin\left(\frac{1}{x}\right) \leq x}$$

Since $\lim_{x \rightarrow 0} -x = 0$ and $\lim_{x \rightarrow 0} x = 0$,

the Squeeze Theorem says that

$$\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) = 0$$

2(a) Suppose one were to solve the equation $\cos x = \sqrt{x}$ using Newton's method. Find a formula for the $(n+1)$ st approximation in terms of the n th approximation.

We want x where $\underbrace{\cos x - \sqrt{x}}_{f(x)} = 0$.

Newton's method says

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

So

$$x_{n+1} = x_n - \frac{\cos(x_n) - \sqrt{x_n}}{-\sin(x_n) - \frac{1}{2\sqrt{x_n}}}$$

(b) Find a function $f(x)$ such that $f''(x) = 2 + \cos x$, $f(0) = -1$, and $f(\pi/2) = 0$.

Antidifferentiating yields:

and $f'(x) = 2x + \sin x + C$

$$f(x) = x^2 - \cos x + Cx + D$$

$f(0) = -1$ gives: $-1 = 0 - 1 + 0 + D$
 $\Rightarrow \underline{D = 0}$

$f(\pi/2) = 0$ gives: $0 = (\pi/2)^2 + 0 + C(\pi/2)$
 $\Rightarrow \underline{C = -\pi/2}$

$$f(x) = x^2 - \cos x - \frac{\pi}{2}x$$

3. Find:

(a) $\lim_{x \rightarrow 16} \frac{\sqrt[4]{x} - 2}{x - 16}$ [Hint: this is the derivative of a function at a point.]

This is $f'(16)$ where $f(x) = x^{\frac{1}{4}}$.

$$\text{So } f'(x) = \frac{1}{4} x^{-3/4}$$

$$f'(16) = \frac{1}{4} (16)^{-3/4} = \frac{1}{4} (16^{\frac{1}{4}})^{-3}$$

$$= \frac{1}{4} (2)^{-3} = \frac{1}{4} \cdot \frac{1}{8} = \boxed{\frac{1}{32}}$$

(b) $\lim_{t \rightarrow 0} \frac{\tan(6t)}{\sin(2t)}$ $= \lim_{t \rightarrow 0} \frac{\sin(6t)}{\sin(2t) \cos(6t)} \cdot \frac{2t}{6t} \cdot \frac{6}{2}$

$$= \lim_{t \rightarrow 0} \underbrace{\frac{\sin(6t)}{6t}}_{\rightarrow 1} \cdot \underbrace{\frac{2t}{\sin(2t)}}_{\rightarrow 1} \cdot \underbrace{\frac{1}{\cos(6t)}}_{\rightarrow 1} \cdot \underbrace{\frac{6}{2}}_{\rightarrow 3}$$

$$= \boxed{3}$$

(c) $F'(5)$ where $F(x) = f(g(x))$, $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$, and $g'(5) = 6$.

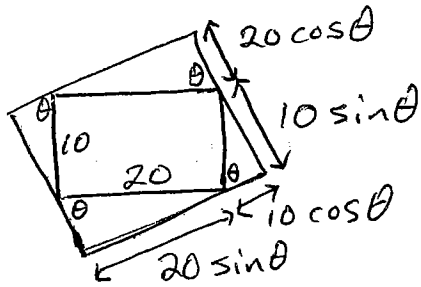
$$F'(x) = f'(g(x)) g'(x)$$

$$F'(5) = f'(g(5)) g'(5)$$

$$= f'(-2) g'(5)$$

$$= \boxed{24}$$

4. Find the maximum area of a rectangle that can be circumscribed about a 10×20 rectangle.
[Express area in terms of an angle θ .]



$$\text{Area} = \text{length} \times \text{width}$$

$$= (20 \sin \theta + 10 \cos \theta)(10 \sin \theta + 20 \cos \theta)$$

$$= 200 \sin^2 \theta + 100 \cos \theta \sin \theta + 400 \sin \theta \cos \theta + 200 \cos^2 \theta$$

$$A(\theta) = 500 \sin \theta \cos \theta + 200$$

Maximize this over $0 \leq \theta \leq \pi/2$.

Crit. Pts: $A'(\theta) = 500(\cos^2 \theta - \sin^2 \theta)$ by product rule

$$= 500(\cos^2 \theta - (1 - \cos^2 \theta))$$

$$= 500(2\cos^2 \theta - 1)$$

$$A'(\theta) = 0 \Rightarrow 2\cos^2 \theta - 1 = 0 \Rightarrow \cos^2 \theta = \frac{1}{2}$$

$$\cos \theta = \frac{\sqrt{2}}{2}$$

$$\theta = \pi/4$$

check $A(\theta)$ at critical and endpoints:

$$A(0) = 200$$

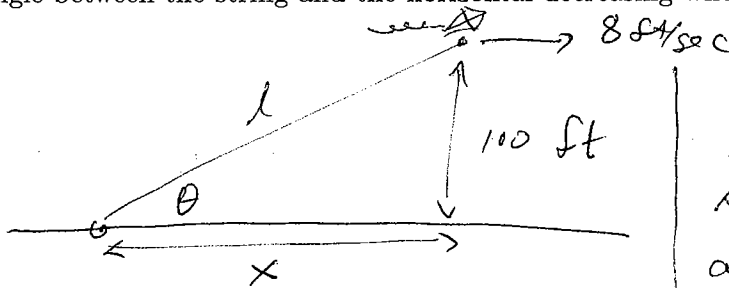
$$A(\pi/2) = 200$$

$$A(\pi/4) = 500 \left(\frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} \right) + 200$$

$$= \frac{500}{2} + 200 = 450$$

So maximum area is 450.

5. A kite 100 ft above the ground moves horizontally at a speed of 8 ft/sec. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string has been let out?



θ , x , and l vary.

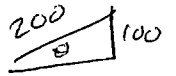
At one instant, $l = 200$
and $\frac{dx}{dt} = 8$. Want $\frac{d\theta}{dt}$.

Use: $\tan \theta = \frac{100}{x}$ (or $\cot \theta = \frac{x}{100}$)
(relation).

take $\frac{d}{dt}$:

$$\sec^2 \theta \frac{d\theta}{dt} = \frac{-100}{x^2} \frac{dx}{dt}$$

Plug in info now. What are x and $\sec \theta$ when $l = 200$?
- when $l = 200$, we have $\cos \theta = \frac{\sqrt{3}}{2}$



So $\sec \theta = \frac{2}{\sqrt{3}}$.

Also, $x^2 + 100^2 = 200^2$

so $x^2 = 30000$.

Plugging in:

$$\left(\frac{2}{\sqrt{3}}\right)^2 \frac{d\theta}{dt} = \frac{-100}{30000} (8)$$

$$\frac{4}{3} \cdot \frac{d\theta}{dt} = \frac{-8}{300}, \Rightarrow \frac{d\theta}{dt} = \frac{-8}{400} = \frac{-1}{50}$$

so angle decreases at rate of $\frac{1}{50}$ radians/sec.

