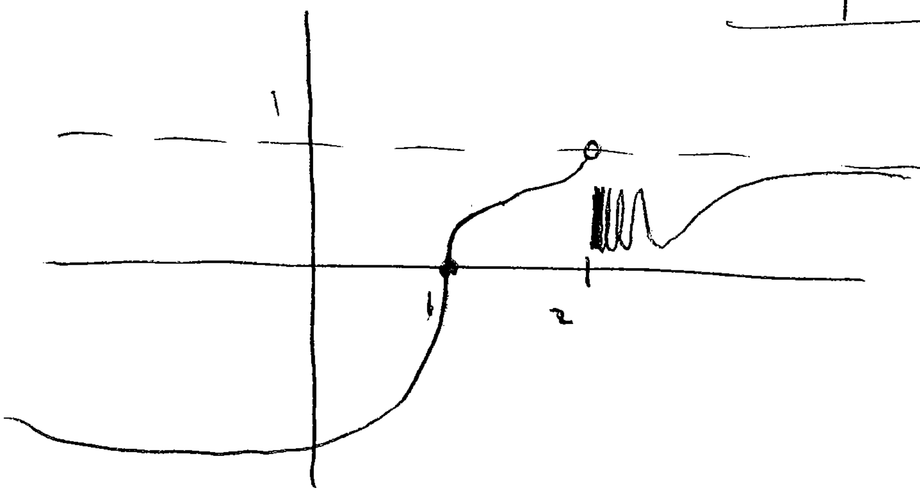
1(a) Draw the graph of a function $f(x)$ satisfying:
$|f(x)| < 1$ for $x > 2$
$\lim_{x\to 2^+} f(x)$ does not exist
$\lim_{x\to 2^-} f(x) = 1$
$\lim_{x\to\infty} f(x) = 1$
$y = f(x)$ has a tangent line at every $x < 2$
$f(x)$ is continuous but not differentiable at $x = 1$

one possibility:



(b) Using a theorem from the course, show that $\lim\limits_{x\to 0} x\sin(1/x) = 0$.

Note that $-1 \le \sin\left(\frac{1}{x}\right) \le 1$ for all x

So $-x \le x\sin\left(\frac{1}{x}\right) \le x$.

Since $\lim\limits_{x\to 0} -x = 0$ and $\lim\limits_{x\to 0} x = 0$,

the Squeeze Theorem says that

$\lim\limits_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$.

2(a) Suppose one were to solve the equation $\cos x = \sqrt{x}$ using Newton's method. Find a formula for the $(n+1)$st approximation in terms of the nth approximation.

We want x where $\underbrace{\cos x - \sqrt{x}}_{= f(x)} = 0$.

Newton's method says

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

so

$$\boxed{x_{n+1} = x_n - \frac{\cos(x_n) - \sqrt{x_n}}{-\sin(x_n) - \frac{1}{2\sqrt{x_n}}}}$$

(b) Find a function $f(x)$ such that $f''(x) = 2 + \cos x$, $f(0) = -1$, and $f(\pi/2) = 0$.

Antidifferentiating yields:

$$f'(x) = 2x + \sin x + C$$

and

$$\underline{f(x) = x^2 - \cos x + Cx + D}$$

$\underline{f(0) = -1}$ gives: $-1 = 0 - 1 + 0 + D$

$$\Rightarrow \underline{D = 0}$$

$\underline{f(\pi/2) = 0}$ gives: $0 = (\pi/2)^2 + 0 + C(\pi/2)$

$$\Rightarrow \underline{C = -\pi/2}$$

$$\boxed{f(x) = x^2 - \cos x - \frac{\pi}{2}x}$$

3. Find:

(a) $\lim_{x\to 16} \frac{\sqrt[4]{x}-2}{x-16}$ [Hint: this is the derivative of a function at a point.]

This is $f'(16)$ where $f(x) = x^{\frac{1}{4}}$.

So $f'(x) = \frac{1}{4}x^{-3/4}$

$f'(16) = \frac{1}{4}(16)^{-3/4} = \frac{1}{4}(16^{\frac{1}{4}})^{-3}$

$= \frac{1}{4}(2)^{-3} = \frac{1}{4}\cdot\frac{1}{8} = \boxed{\frac{1}{32}}$

(b) $\lim_{t\to 0} \frac{\tan(6t)}{\sin(2t)}$ $= \lim_{t\to 0} \frac{\sin(6t)}{\sin(2t)\cos(6t)} \cdot \frac{2t}{6t} \cdot \frac{6}{2}$

$= \lim_{t\to 0} \underbrace{\frac{\sin(6t)}{6t}}_{\to 1} \cdot \underbrace{\frac{2t}{\sin(2t)}}_{\to 1} \cdot \underbrace{\frac{1}{\cos(6t)}}_{\to 1} \cdot \underbrace{\frac{6}{2}}_{\to 3}$

$= \boxed{3}$

(c) $F'(5)$ where $F(x) = f(g(x))$, $f(-2) = 8$, $f'(-2) = 4$, $f'(5) = 3$, $g(5) = -2$, and $g'(5) = 6$.

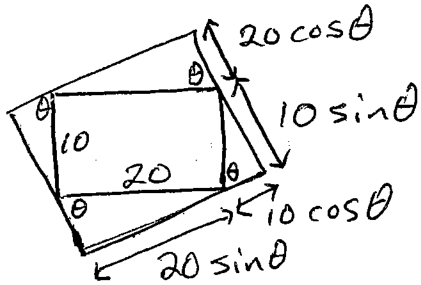
$F'(x) = f'(g(x))g'(x)$

$F'(5) = f'(g(5))g'(5)$

$= f'(-2)g'(5)$

$= \boxed{24}$

4. Find the maximum area of a rectangle that can be circumscribed about a 10×20 rectangle. [Express area in terms of an angle θ.]



Area = length × width

$$= (20\sin\theta + 10\cos\theta)(10\sin\theta + 20\cos\theta)$$

$$= 200\sin^2\theta + 100\cos\theta\sin\theta + 400\sin\theta\cos\theta + 200\cos^2\theta$$

$$A(\theta) = 500\sin\theta\cos\theta + 200$$

Maximize this over $0 \le \theta \le \pi/2$.

Crit. Pts: $A'(\theta) = 500(\cos^2\theta - \sin^2\theta)$ by product rule

$$= 500(\cos^2\theta - (1-\cos^2\theta))$$

$$= 500(2\cos^2\theta - 1).$$

$A'(\theta) = 0 \Rightarrow 2\cos^2\theta - 1 = 0 \Rightarrow \cos^2\theta = \frac{1}{2}$

$\cos\theta = \frac{\sqrt{2}}{2}$

$\theta = \pi/4$.

Check $A(\theta)$ at critical and endpoints:

$A(0) = 200$

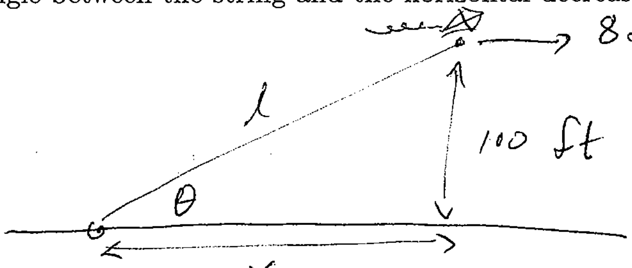
$A(\pi/2) = 200$

$A(\pi/4) = 500\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + 200$

$= \frac{500}{2} + 200 = 450.$

$\boxed{\text{So maximum area is } 450}$.

5. A kite 100 ft above the ground moves horizontally at a speed of 8 ft/sec. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string has been let out?



θ, x, and l vary.

At one instant, $l = 200$ and $\frac{dx}{dt} = 8$. Want $\frac{d\theta}{dt}$.

Use: $\tan\theta = \frac{100}{x}$ (or $\cot\theta = \frac{x}{100}$)

(relation).

take $\frac{d}{dt}$:

$$\sec^2\theta \frac{d\theta}{dt} = \frac{-100}{x^2}\frac{dx}{dt}$$

Plug in info now. What are x and $\sec\theta$ when $l = 200$?

- when $l = 200$, we have $\cos\theta = \frac{\sqrt{3}}{2}$

So $\sec\theta = \frac{2}{\sqrt{3}}$.

Also, $x^2 + 100^2 = 200^2$

so $x^2 = 30000$.

Plugging in:

$$\left(\frac{2}{\sqrt{3}}\right)^2 \frac{d\theta}{dt} = \frac{-1\cancel{00}}{300\cancel{00}}(8)$$

$$\frac{4}{3}\cdot\frac{d\theta}{dt} = \frac{-8}{300}, \Rightarrow \frac{d\theta}{dt} = \frac{-8}{400} = \frac{-1}{50}.$$

$\boxed{\text{so angle decreases at rate of } \frac{1}{50} \text{ radians/sec.}}$

6(a) Find an equation of the tangent line to the curve $y^3 + yx^2 + x^2 - 3y^2 = 0$ at the point $(0, 3)$.

take $\frac{d}{dx}$ of both sides:

$$3y^2 \frac{dy}{dx} + y(2x) + x^2 \frac{dy}{dx} + 2x - 6y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(3y^2 + x^2 - 6y) + 2xy + 2x = 0$$

$$\frac{dy}{dx} = \frac{-2xy - 2x}{3y^2 + x^2 - 6y}.$$

At $(0,3)$: $\quad \frac{dy}{dx} = \frac{0}{27-18} = 0 =$ slope of tang. line.

slope 0, through $(0,3)$:

T. Line is $\boxed{y = 3}$

(b) Find all critical points of the function $f(x) = x^5 - 2x^4 + x^3$. What does the second derivative test say about each of these points?

$$f'(x) = 5x^4 - 8x^3 + 3x^2 = x^2(5x^2 - 8x + 3)$$
$$= x^2(5x-3)(x-1).$$

So $f'(x) = 0$ when

$\boxed{x = 0,\ x = 1,\ x = \frac{3}{5}}$

critical points.

$$f''(x) = 20x^3 - 24x^2 + 6x.$$

test:

$f''(0) = 0 \rightarrow$ no information about $x=0$.

$f''(1) = 2 \rightarrow$ local minimum at 1

$$f''\left(\tfrac{3}{5}\right) = 20\left(\tfrac{27}{125}\right) - 24\left(\tfrac{9}{25}\right) + 6\left(\tfrac{3}{5}\right)$$
$$= \frac{4 \cdot 27}{25} - \frac{24 \cdot 9}{25} + \frac{6 \cdot 15}{25} = \frac{108 - 216 + 90}{25}$$
$$= \frac{-18}{25} < 0.$$

$\rightarrow$ $\boxed{\text{local maximum at } 3/5.}$