1(a) (5 points) Find the absolute maximum and absolute minimum values of $f(x) = x^4 - 2x^2 + 3$ on the interval $[-2, 3]$.

$f'(x) = 4x^3 - 4x = 0$, $\quad 4x(x^2-1) = 0$, $\quad \underline{x = 0, \pm 1}$.

critical points

check endpoints also:

$f(-2) = 11$

$f(-1) = 2$

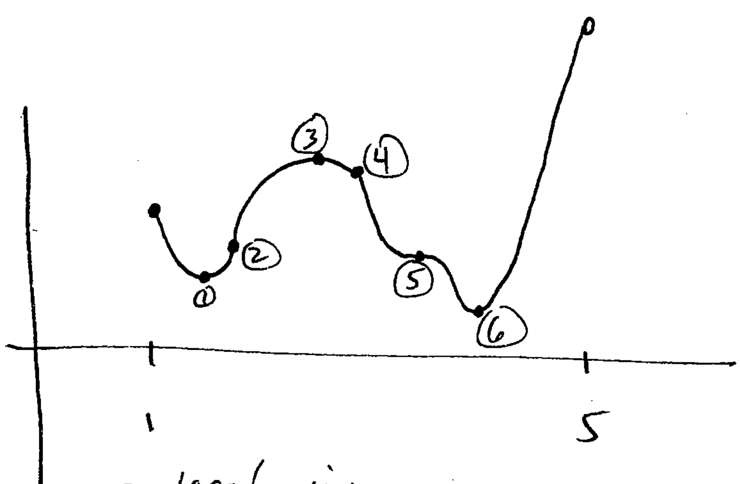
$f(0) = 3$

$f(1) = 2$

$f(3) = 66$

$\boxed{\text{absolute max. is } 66 \quad \text{absolute min. is } 2}$.

(b) (5 points) Sketch the graph of a function f that is continuous on $[1, 5)$ and has two local minima, one local maximum, no absolute maximum, and six critical numbers.



① local min

② vertical tangent

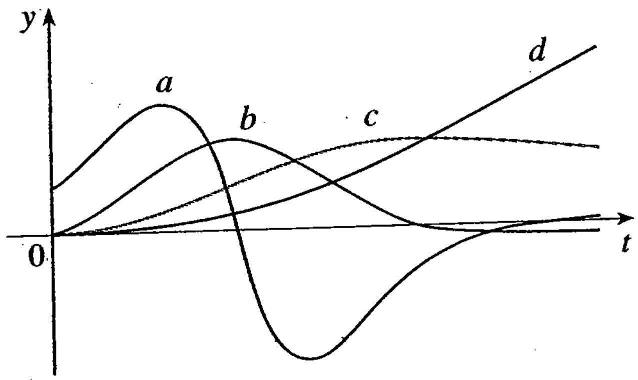
③ local max

④ corner (f' does not exist)

⑤ $f' = 0$

⑥ local min

2(a) (5 points) The figure shows the graphs of four functions. Out of the four, identify three of them that could represent the position, velocity, and acceleration of a car (and say which is which).



c = position
b = velocity
a = acceleration

OR

d = position
c = velocity
b = acceleration

(b) (5 points) Find the equation of the tangent line to the ellipse $x^2 + xy + y^2 = 3$ at the point $(1, 1)$. [Hint: use implicit differentiation.]

$$2x + \left(x\frac{dy}{dx} + y\right) + 2y\frac{dy}{dx} = 0$$

(diff. both sides w.r.t. x)

$$2x + y = (-2y - x)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x+y}{-2y-x}$$ at (1,1) slope is $\frac{2(1)+1}{-2(1)-1}$

$= -1.$

tangent line:

$$\boxed{(y-1) = -1(x-1)}$$

3. (10 points) Let $f(x) = x^{2/3}$.

(a) Find the linearization of $f(x)$ at the point $a = 8$.

$$f'(x) = \frac{2}{3} x^{-1/3}$$

$$f'(8) = \frac{2}{3}(8)^{-1/3} = \frac{1}{3}$$

$f(8) = 4$ tang. line : $y - 4 = \frac{1}{3}(x - 8)$

$$y = \frac{1}{3}x + \frac{4}{3}$$

$$\boxed{L(x) = \frac{1}{3}x + \frac{4}{3}}$$

(b) Estimate $(7.9993)^{2/3}$.

$$(7.9993)^{2/3} \approx L(7.9993)$$

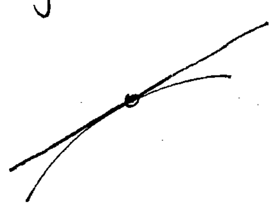
$$= \frac{7.9993}{3} + \frac{4}{3} = \boxed{\frac{11.9993}{3}}$$

$$= \boxed{3.9997666\ldots}$$

Extra credit Using $f''(x)$, can you say whether your estimate in (b) is an over- or under-estimate? Give an explanation with a picture.

$f''(x) = \frac{-2}{9} x^{-4/3}$, $f''(8) = \frac{-2}{9}(8)^{-4/3} < 0$

Since $f''(8) < 0$, graph of $y = x^{2/3}$ is concave down, so tangent line is above the graph:



So $L(7.9993)$ is an $\boxed{\text{overestimate}}$.

4. (10 points)

(a) State the Mean Value Theorem.

If f is continuous on $[a, b]$ and differentiable on (a, b) then there is a point c in (a, b) where

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

(b) If $f(0) = 4$ and $f'(x) \geq 6$ for $0 \leq x \leq 10$, what can you say about $f(10)$? Explain your (mathematical) reasoning.

Using MVT, with $a=0$, $b=10$. At some c between 0 and 10, $f'(c) = \frac{f(10)-4}{10-0}$

and $f'(c) \geq 6$.

So $\frac{f(10)-4}{10} \geq 6$

$$\Rightarrow f(10)-4 \geq 60 \quad \Rightarrow \boxed{f(10) \geq 64}$$

(c) Give a non-mathematical explanation using a driving analogy.

If you drive along a road for 10 hours and your speed never drops below 6 mph, then you will travel at least 60 miles.

5(a) (6 points) Use implicit differentiation to find $\frac{dy}{dx}$ for each of the curves $y = ax^3$ and $x^2+3y^2 = b$.

$y = ax^3$

$\boxed{\frac{dy}{dx} = 3ax^2}$

$x^2 + 3y^2 = b$

$2x + 6y\frac{dy}{dx} = 0$

$\boxed{\frac{dy}{dx} = \frac{-2x}{6y}}$

(b) (4 points) Show that the curves $y = ax^3$ and $x^2 + 3y^2 = b$ are orthogonal. [Hint: Use $y = ax^3$ to eliminate y from your expressions found in part (a).]

At any point (x,y) the curve $y=ax^3$ has slope $3ax^2$ and the curve $x^2+3y^2=b$ has slope

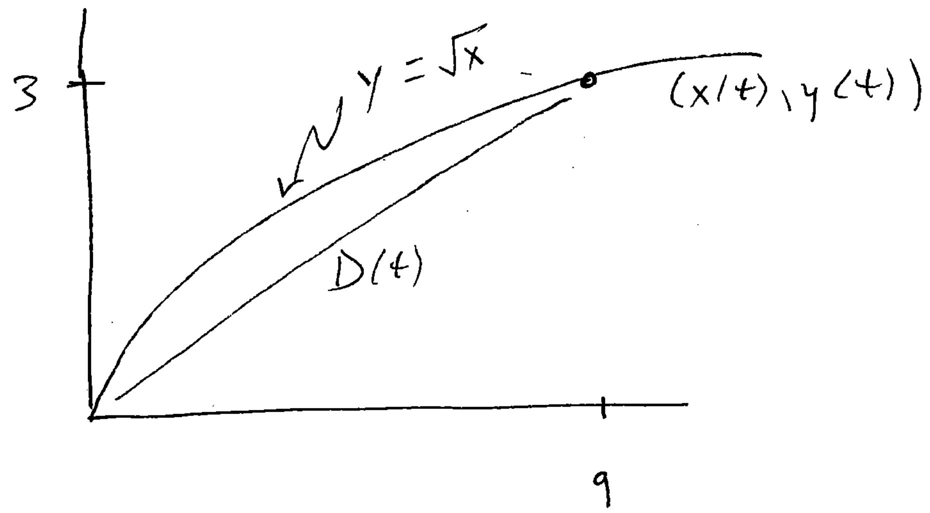
$$\frac{-2x}{6y} = \frac{-2x}{6ax^3} = \frac{-1}{3ax^2}.$$

The product of the two slopes is

$$(3ax^2)\left(\frac{-1}{3ax^2}\right) = -1$$

so the curves are orthogonal.

6. (10 points) A particle is moving alone the curve $y = \sqrt{x}$. As the particle passes through the point $(9, 3)$, its x-coordinate is increasing at a rate of 4 units per second. How fast is the distance from the particle to the origin changing at this instant?



Let $D(t)$ = distance from origin.

Then $D(t)^2 = x(t)^2 + y(t)^2$

So $\underline{2D(t)D'(t) = 2x(t)x'(t) + 2y(t)y'(t)}$.

At the given time we have

$x = 9, \quad y = 3, \quad D = \sqrt{90}, \quad x' = 4.$

From $y(t) = \sqrt{x(t)}$ we get $\underline{y'(t) = \frac{1}{2\sqrt{x(t)}}x'(t)}$

So $y' = \frac{1}{6}(4) = \frac{2}{3}$.

Putting everything in we get:

$$2\sqrt{90}\, D'(t) = 2(9)(4) + 2(3)\left(\frac{2}{3}\right)$$

$$\boxed{D'(t) = \frac{38}{\sqrt{90}} \text{ units per second}}$$