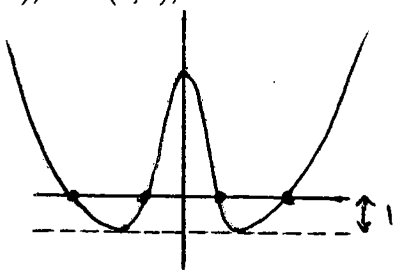
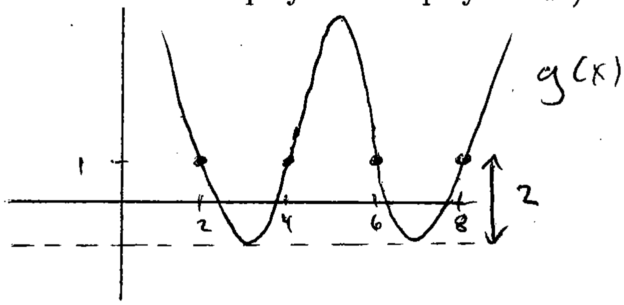
1(a) (5 points) The graph of $f(x) = x^4 - 4x^2 + 3$ passes through the points $(-3,0)$, $(-1,0)$, $(1,0)$ and $(3,0)$. Its minimum value (attained at $x = \pm\sqrt{2}$) is -1.

Transform $f(x)$ to obtain a new degree 4 polynomial that passes through the points $(2,1)$, $(4,1)$, $(6,1)$, and $(8,1)$, and has minimum value -1. (You do not need to simplify the new polynomial.)





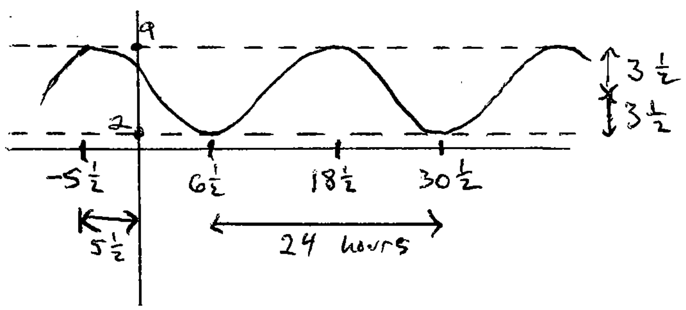
To get $g(x)$ we:

- stretch vertically by 2: $2x^4 - 8x^2 + 6$
- shift up by 1: $2x^4 - 8x^2 + 7$
- shift right by 5:

$$g(x) = 2(x-5)^4 - 8(x-5)^2 + 7$$

(b) (5 points) The water in a tidal basin varies in depth from 2 feet to 9 feet deep. Low tide is at 6:30 am and high tide is at 6:30 pm. Write down a function $d(t)$ giving the depth at time t, where t is the number of hours since midnight.

Here is a graph:



To get from $\cos(t)$ we:

- stretch vertically by $3\frac{1}{2}$

$3\frac{1}{2}\cos t$

- shift up by $5\frac{1}{2}$

$3\frac{1}{2}\cos t + 5\frac{1}{2}$

- stretch horizontally by $\frac{24}{2\pi}$

$3\frac{1}{2}\cos\left(\frac{2\pi}{24}t\right) + 5\frac{1}{2}$

- shift left by $5\frac{1}{2}$

$$d(t) = 3\tfrac{1}{2}\cos\left(\frac{2\pi}{24}\left(t + 5\tfrac{1}{2}\right)\right) + 5\tfrac{1}{2}$$

2. (10 points) Find each of the following limits (or say that they do not exist). Give brief explanations.

(a) $\lim_{x\to 1} \frac{x^2-1}{x^2-2x+1}$ $= \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)(x-1)} = \lim_{x\to 1} \frac{x+1}{x-1}$ since these functions agree away from 1

The latter limit does not exist - the quantity approaches ∞.

(b) $\lim_{h\to 0} \frac{(h-1)^3+1}{h}$ $= \lim_{h\to 0} \frac{h^3-3h^2+3h-1+1}{h}$

$= \lim_{h\to 0} h^2-3h+3 = 3$

(c) $\lim_{x\to 4^-} \frac{|x-4|}{x-4}$ The function is $f(x) = \begin{cases} -1 & x<4 \\ 1 & x>4 \end{cases}$.

$=$

$\lim_{x\to 4^-} -1$ since -1 and $f(x)$ agree to the left of 4

$= -1$

(d) $\lim_{x\to -1} f(x)$ where $-2x \le f(x) \le x^2+1$ for all x satisfying $-2 < x < 0$

$\lim_{x\to -1} -2x = 2$ and $\lim_{x\to -1} x^2+1 = 2$, so

$\lim_{x\to -1} f(x) = 2$ by the Squeeze Theorem.

(e) $\lim_{t\to -1} (t^2+1)^3(t+3)^5$

$= ((-1)^2+1)^3(-1+3)^5$

$= 2^3 2^5$

$= 2^8$ by direct substitution

3. (10 points) The displacement (in meters) of an object moving in a straight line is given by $s = 4 + 2t^2$, where t is measured in seconds.

(a) Find the average velocity over the time period $[1, T]$ where $T > 1$. Explain what you are doing.

Avg. velocity is $\frac{\text{change in displacement}}{\text{amount of time}} = \frac{s(T) - s(1)}{T-1}$

$$= \frac{4 + 2T^2 - (4 + 2)}{T - 1}$$

$$= \frac{2T^2 - 2}{T-1} = \frac{2(T-1)(T+1)}{T-1} = 2(T+1) \text{ meters per second}$$

(b) Find the instantaneous velocity when $t = 1$. Explain what you are doing.

Instantaneous velocity = limit of avg. vel. over $[1, T]$ as $T \to 1$

$$= \lim_{T \to 1} 2(T+1)$$

$= 4$ meters per second.

4. (10 points) Let $f(x) = |x - 2|x|| + 2$.

(a) Write a new expression for $f(x)$ without using absolute value symbols.

Working from inside out:

$$f(x) = \begin{cases} |x - 2x| + 2 & x \geq 0 \\ |x - 2(-x)| + 2 & x < 0 \end{cases}$$

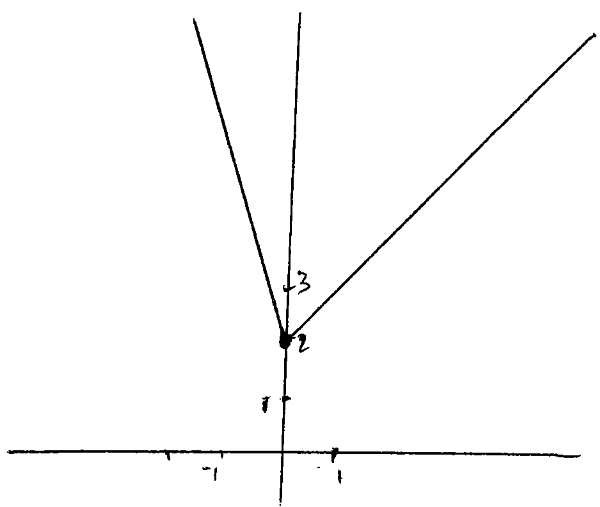
$$= \begin{cases} |-x| + 2 & x \geq 0 \\ |3x| + 2 & x < 0 \end{cases}$$

$$= \begin{cases} x + 2 & x \geq 0 \\ -3x + 2 & x < 0 \end{cases}$$

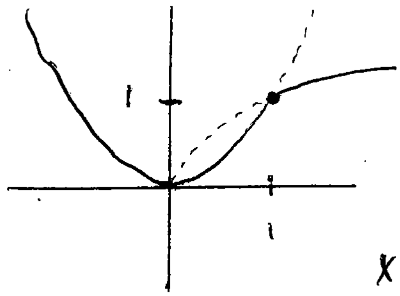
since $-x$ is negative when x is positive

since $3x$ is negative when x is negative

(b) Carefully draw the graph of $f(x)$, labelling any significant points.



5(a) (5 points) Let $f(x) = \begin{cases} x^2 & x < 1 \\ \sqrt{x} & x \geq 1 \end{cases}$. Draw the graph and explain carefully why $f(x)$ is continuous at $x = 1$. Is it continuous everywhere?



Want: $\lim_{x\to 1} f(x) = 1 \; (= f(1))$

do one sided limits:

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \sqrt{x} = 1$

they agree, so $\lim_{x\to 1} f(x) = 1$.

Yes it's continuous everywhere, since x^2 and $\sqrt{x}$ are.

(b) (5 points) Carefully state the Intermediate Value Theorem. Then show that there is a number x between 0 and 1 that solves the equation $2\sin x = 2 - 3x$. [Hint: rearrange the equation.]

IVT: If $f(x)$ is continuous on $[a, b]$ and N is a number between $f(a)$ and $f(b)$, then there is a number x between a and b with $f(x) = N$.

Here, let $f(x) = 2\sin x - 2 + 3x$. Then $f(0) = -2$, $f(1) = 2\sin(1) + 1$, which is positive.

Take $N = 0$ in the IVT, and $a = 0$, $b = 1$.

$\Rightarrow f(x) = 0$ for some x between 0 and 1.

i.e. $2\sin x - 2 + 3x = 0$, i.e. $2\sin x = 2 - 3x$.

6(a) (5 points) Draw the graph of a function $f(x)$ satisfying all of these properties:

(i) $|f(x)| < 1$ for all $x \leq 0$

(ii) $\lim_{x \to 0^-} f(x)$ does not exist

(iii) $\lim_{x \to 0^+} f(x) = f(0) = 0$

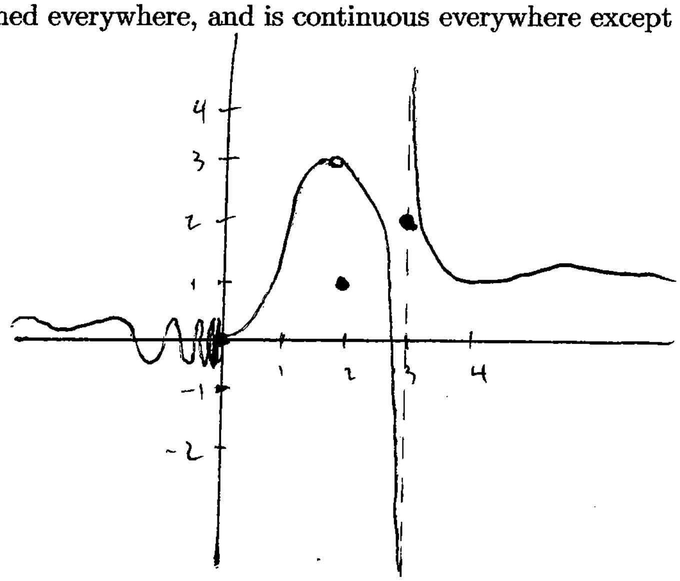
(iv) $\lim_{x \to 2} f(x) = 3$

(v) $f(2) = 1$

(vi) $\lim_{x \to 3^-} f(x) = -\infty$

(vii) $\lim_{x \to 3^+} f(x) = +\infty$

(viii) $f(x)$ is defined everywhere, and is continuous everywhere except at $x = 0, 2, 3$



(b) (5 points) For each of the locations $x = 0, 2, 3$ say whether your function $f(x)$ has a removable, jump, or infinite discontinuity (or none of these). Also indicate (for each location) whether $f(x)$ is continuous from the left and/or from the right.

$\underline{x=0}$ none of these; continuous from the right only

$\underline{x=2}$ removable discontinuity; not continuous from either side

$\underline{x=3}$ infinite discontinuity; not continuous from either side