1. (10 points) Analyze the function $f(x) = \frac{x-1}{x^2}$, by determining its noteworthy features and where they occur, and use this information to sketch a detailed graph of $f(x)$. (What is the domain? Where is it positive/negative? increasing/decreasing? concave up/down? Are there asymptotes?) Provide the coordinates of any interesting points on the curve.

Domain: $x \neq 0$.

positive: $x > 1$ negative: $x < 1$

$$f'(x) = \frac{x^2 - (x-1)2x}{x^4} = \frac{2x - x^2}{x^4} = \frac{2-x}{x^3}.$$

increasing: $x > 0$ and $x < 2$, i.e. $(0, 2)$

decreasing: $(-\infty, 0)$, $(2, \infty)$

$$f''(x) = \frac{x^3(-1) - (2-x)3x^2}{x^6} = \frac{2x^3 - 6x^2}{x^6} = \frac{2x-6}{x^4}.$$

concave up: $(3, \infty)$

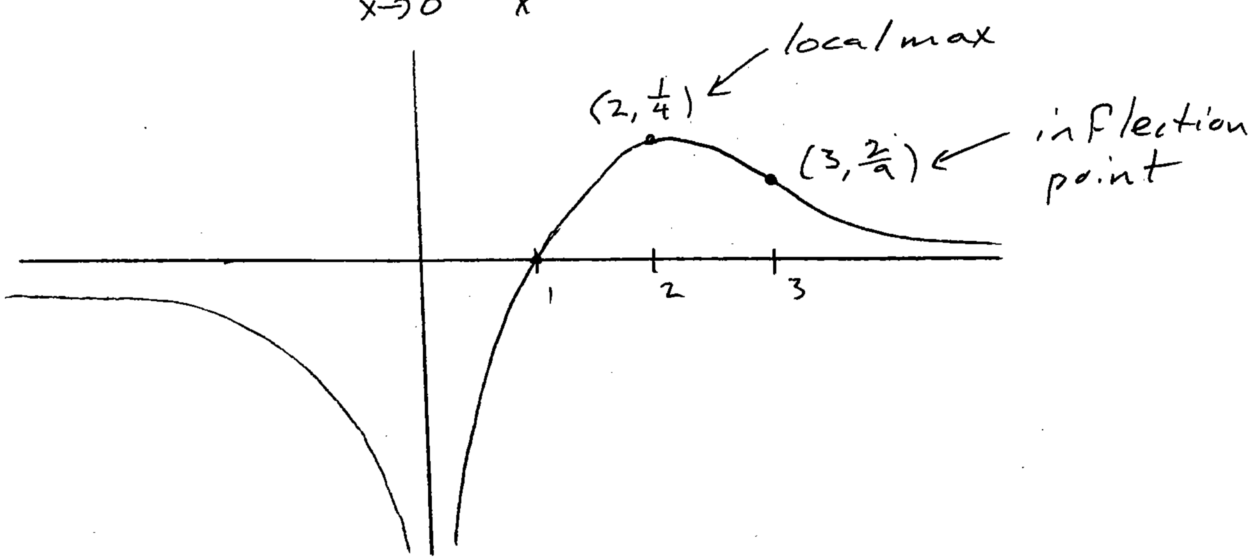
concave down: $(-\infty, 0)$, $(0, 3)$

$f(1) = 0$, $f(2) = \frac{1}{4}$, $f(3) = \frac{2}{9}$.

Asymptotes? $\lim_{x \to \infty} \frac{x-1}{x^2} = \lim_{x \to \infty} \frac{1}{x} - \frac{1}{x^2} = 0$

$\lim_{x \to -\infty} \frac{x-1}{x^2} = \lim_{x \to -\infty} \frac{1}{x} - \frac{1}{x^2} = 0$

$\lim_{x \to 0} \frac{x-1}{x^2} = -\infty$



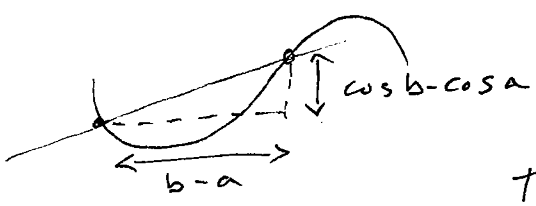
2. (15 points) We wish to solve the equation $\cos x = 2x - 3$, or equivalently, $f(x) = 0$ where $f(x) = \cos x - 2x + 3$.

(a) Use the Intermediate Value Theorem to explain why there is at least one solution. Be sure to check that all requirements of the theorem are met.

$f(x)$ is continuous and $f(-\pi/2) = \pi + 3 > 0$ and $f(\frac{3\pi}{2}) = -3\pi + 3 < 0$ so the IVT says that $f(c) = 0$ for some number c between $-\pi/2$ and $3\pi/2$.

(b) Use the Mean Value Theorem to show that the graph of $y = \cos x$ and the line $y = 2x - 3$ cannot meet in more than one point.

If they meet at 2 points, say at $x = a$ and $x = b$, then $\dfrac{\cos(b) - \cos(a)}{b - a}$ = slope of the line $= 2$.



The MVT says that the derivative of $\cos x$ must be 2 somewhere between a and b. But $\frac{d}{dx}(\cos x) = -\sin x$, and this is always between 1 and -1. So this situation cannot occur.

(c) Now we know that the equation has exactly one solution. Suppose we were to find it using Newton's method. Find a formula for the $(n+1)$-st approximation x_{n+1} in terms of x_n. (Your formula should be specific to this example.) What would be a good initial guess x_1?

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\boxed{x_{n+1} = x_n - \frac{\cos(x_n) - 2x_n + 3}{-\sin(x_n) - 2}}$$

3. (5 points) The figure below shows the graph $y = f(x)$. Suppose we were to use Newton's method to try to find the roots of $f(x)$. Find x-values a, b, and c, so that:

(a) if $x_1 = a$ is the initial guess, the sequence (x_n) converges to the root $x = 2$

(b) if $x_1 = b$ is the initial guess, the sequence goes to infinity

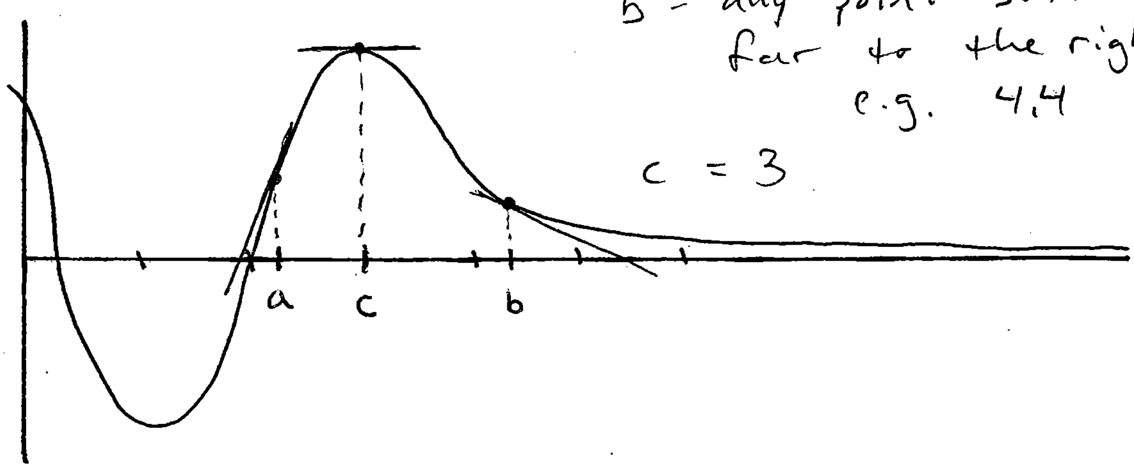
(c) if $x_1 = c$ is the initial guess, the method stops and cannot be continued.

Also indicate these points on the graph.

a = any point close to 2, e.g. 2.3

b = any point sufficiently far to the right, e.g. 4.4

c = 3



4. (10 points) Calculate each of the following.

(a) $\frac{dw}{dz}$ if $\tan(z/w) = z + w$

take $\frac{d}{dz}$ of both sides:

$$\sec^2(z/w)\left(\frac{w - z\frac{dw}{dz}}{w^2}\right) = 1 + \frac{dw}{dz}$$

$$\sec^2(z/w)\left(w - z\frac{dw}{dz}\right) = w^2 + w^2\frac{dw}{dz}$$

$$\frac{dw}{dz}\left(w^2 + z\sec^2(z/w)\right) = w\sec^2(z/w) - w^2$$

$$\boxed{\frac{dw}{dz} = \frac{w\sec^2(z/w) - w^2}{w^2 + z\sec^2(z/w)}}$$

(b) the derivative of $\cos(\cos(\sin(x)))\cos(\sin(\sin(x)))$

$$\cos(\cos(\sin x))(-\sin(\sin(\sin x)))\frac{d}{dx}(\sin(\sin x)) +$$
$$- \sin(\cos(\sin x))\frac{d}{dx}(\cos(\sin x))\cos(\sin(\sin x))$$

$$= \boxed{\begin{aligned}&-\cos(\cos(\sin x))\sin(\sin(\sin x))\cos(\sin x)\cos x \\ &+ \sin(\cos(\sin x))\sin(\sin x)(\cos x)\cos(\sin(\sin x))\end{aligned}}$$

(c) $h'(x)$, if $h(x) = f(1/f(x))$ and $f'(x) = 1/x$

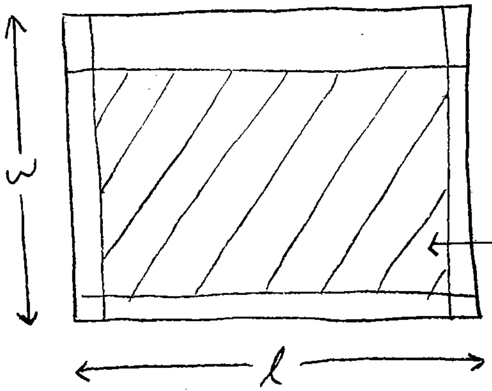
$$h'(x) = f'\left(\frac{1}{f(x)}\right)\frac{d}{dx}\left(\frac{1}{f(x)}\right)$$

$$= f'\left(\frac{1}{f(x)}\right)\left(\frac{-f'(x)}{(f(x))^2}\right)$$

$$= f(x)\left(\frac{-1}{x(f(x))^2}\right)$$

$$= \boxed{\frac{-1}{x f(x)}}$$

5. (10 points) A poster is to have an area of 180 in^2 with 1-inch margins at the bottom and sides and a 2-inch margin at the top. What dimensions will give the largest printed area?



Total area $= 180 \text{ in}^2 = lw$

$\Rightarrow \quad w = \frac{180}{l}$.

Area $= (l-2)(w-3)$

So $A(l) = (l-2)\left(\frac{180}{l} - 3\right)$

$= 180 - \frac{360}{l} - 3l + 6$

$\underline{A(l) = -3l + 186 - \frac{360}{l}}$.

Domain: $l > 0$. Want to maximize $A(l)$.

$A'(l) = -3 + \frac{360}{l^2} = 0$

$-3l^2 + 360 = 0$

$l^2 = 120$

$l = \sqrt{120}$.

<u>Note</u>: if $l < \sqrt{120}$, then $l^2 < 120$, so $\frac{360}{l^2} > 3$

$\Rightarrow A'(l) > 0$.

if $l > \sqrt{120}$, then $A'(l) < 0$

First derivative test $\Rightarrow$ $A(l)$ has abs. maximum at $l = \sqrt{120}$.

Largest printed area when: $\boxed{l = \sqrt{120} \text{ in}, \quad w = \frac{180}{\sqrt{120}} \text{ in}.}$

6(a) (5 points) Find the absolute maximum and absolute minimum of the function $f(x) = \frac{x^2-4}{x^2+4}$ on the interval $[-4, 4]$.

critical numbers: $f'(x) = \dfrac{(x^2+4)2x - (x^2-4)2x}{(x^2+4)^2} = 0$

$2x^3 + 8x - 2x^3 + 8x = 0$

$16x = 0 \Rightarrow x = 0.$

check $x=0$ and endpoints $-4, 4$:

$f(0) = -1$, $f(-4) = \frac{12}{20}$, $f(4) = \frac{12}{20}$.

absolute maximum: $\frac{12}{20}$
absolute minimum: -1

6(b) (5 points) The graph of the *derivative* of a continuous function $f(x)$ is shown.

(i) On what intervals is f increasing or decreasing?

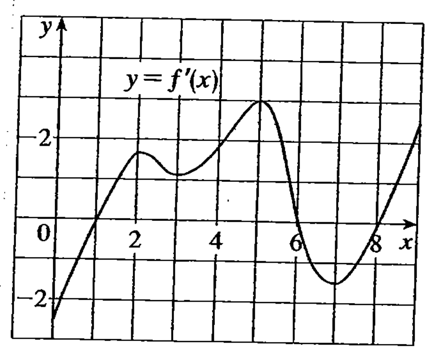
increasing on $(1,6)$, $(8,\infty)$ decreasing on $(-\infty,1)$, $(6,8)$

(ii) At what values of x does f have a local maximum or minimum?

local min at $x=1,8$ local max at $x=6$

(iii) On what intervals is f concave upward or downward?

concave up on $(-\infty,2)$, $(3,5)$, $(7,\infty)$
concave down on $(2,3)$, $(5,7)$



7. (10 points) Find the following limits, and explain your reasoning.

(a) $\lim_{x\to\infty} \dfrac{4x^3 - x + 15}{3x^3 - 4x^2 + 4}$

$$= \lim_{x\to\infty} \frac{4 - \frac{x}{x^3} + \frac{15}{x^3}}{3 - \frac{4x^2}{x^3} + \frac{4}{x^3}}$$

$$= \frac{4 - \lim_{x\to\infty} \frac{1}{x^2} + \lim_{x\to\infty} \frac{15}{x^3}}{3 - \lim_{x\to\infty} \frac{4}{x} + \lim_{x\to\infty} \frac{4}{x^3}}$$

$$= \boxed{\frac{4}{3}}$$

(b) $\lim_{x\to 2} \dfrac{x^2 + x - 6}{x - 2}$

$$= \lim_{x\to 2} \frac{(x-2)(x+3)}{x-2}$$

$$= \lim_{x\to 2} x + 3$$

$$= \boxed{5}$$

(c) $\lim_{x\to 0} \dfrac{\sin 4x}{\sin 6x}$

$$= \lim_{x\to 0} \frac{\sin 4x}{4x} \cdot \frac{6x}{\sin 6x} \cdot \frac{4}{6}$$

$$= \left(\lim_{x\to 0} \frac{\sin 4x}{4x}\right)\left(\lim_{x\to 0} \frac{6x}{\sin 6x}\right) \frac{4}{6}$$

$$= \left(\lim_{4x\to 0} \frac{\sin 4x}{4x}\right)\left(\frac{1}{\lim_{6x\to 0} \frac{\sin 6x}{6x}}\right) \frac{4}{6}$$

$$= (1)(1)\frac{4}{6} = \boxed{\frac{2}{3}}$$

8. (10 points) Solve the following related rates problem. Gravel is being dumped from a conveyor belt at a rate of 30 ft^3/min, and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?

let h = height

then area of base $= \pi \left(\frac{h}{2}\right)^2$

Volume $= \frac{1}{3} h \pi \left(\frac{h}{2}\right)^2 = \frac{\pi}{12} h^3$.

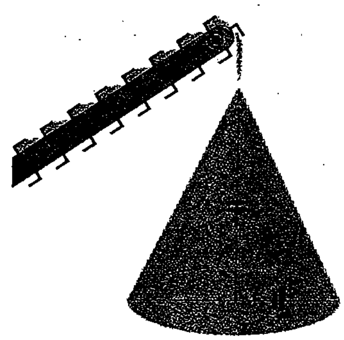
Now diff. both sides w.r.t. t:

$$\underline{V'(t) = \frac{\pi}{4} (h(t))^2 h'(t)}$$

Plug in details:

$$30 = \frac{\pi}{4} (100)\, h'(t)$$

$$h'(t) = \frac{120}{100\pi} = \boxed{\frac{6}{5\pi} \text{ ft/min.}}$$



9(a) (6 points) Sketch the graph of a function that satisfies the given conditions.

(i) $f(0) = 0$, $f'(-2) = f'(1) = f'(9) = 0$,

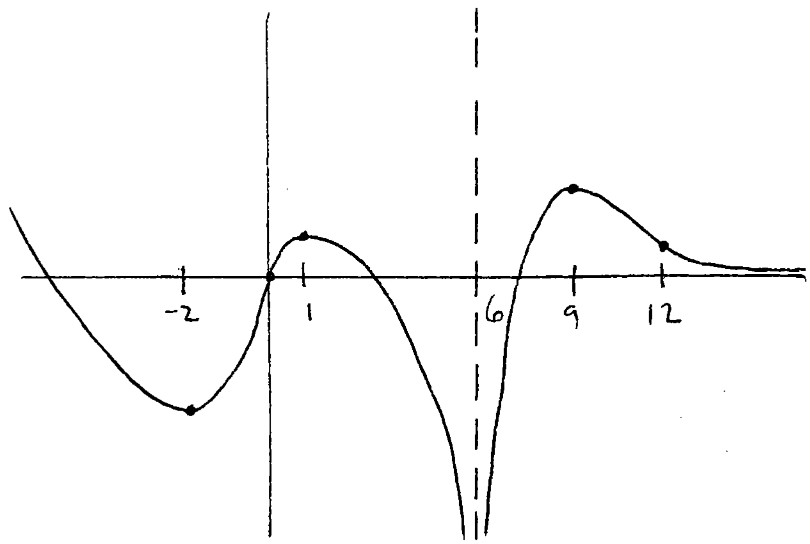
(ii) $\lim_{x\to\infty} f(x) = 0$, $\lim_{x\to 6} = -\infty$,

(iii) $f'(x) < 0$ on $(-\infty, -2)$, $(1, 6)$, and $(9, \infty)$,

(iv) $f'(x) > 0$ on $(-2, 1)$ and $(6, 9)$,

(v) $f''(x) > 0$ on $(-\infty, 0)$ and $(12, \infty)$,

(vi) $f''(x) < 0$ on $(0, 6)$ and $(6, 12)$



(b) (4 points) True or false? No explanations are required.

(i) If $f(x)$ is a polynomial then $\lim_{x\to r} f(x) = f(r)$. True

(ii) If f has an absolute minimum value at c then $f'(c) = 0$. False

(iii) If $f'(x) = g'(x)$ for $0 \le x \le 1$ then $f(x) = g(x)$ for $0 \le x \le 1$. False

(iv) If c is a critical number of a continuous function f and $f''(x) > 0$ for all x, then f has an absolute minimum at c.

True