The Prisoner’s Dilemma
University of Iowa Math Club

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Game theory: mathematical modeling of conflict scenarios

1. 2 or more players (individual, company, nation, species)
2. each player has at least two strategies, courses of possible action
3. the chosen strategies determine the outcome of the game (maybe with chance element)
4. each outcome has a payoff representing the value of the outcome to each player
Definition

A *pure strategy* is a complete description of the choices a player will make during the course of the game. A *payoff matrix* tabulates the results of when pure strategy $x$ is played against pure strategy $y$.

Example (Penny-matching)

Each player chooses H or T. If they match, Rose wins a dollar. If they don’t, Colin wins a dollar.

\[
\begin{array}{cc}
Colin & \\
H & (1, -1) \quad (1, -1) \\
T & (-1, 1) \quad (1, -1) \\
\end{array}
\]

\[
\begin{array}{cc}
Rose & \\
H & (1, -1) \quad (-1, 1) \\
T & (-1, 1) \quad (1, -1) \\
\end{array}
\]
Definition

A *mixed strategy* is a convex combination of pure strategies.

Example (Penny-matching again)

If Rose always plays the same, then Colin always plays the other. Expected payoff: 1 to Colin (−1 to Rose).

If Rose randomizes and plays $H$ half the time, then Colin cannot guarantee more for himself than

$$\frac{1}{2}(1) + \frac{1}{2}(-1) = 0.$$
Mathematical modeling of conflict scenarios

The Prisoner’s Dilemma

Evolutionarily Stable Strategies

Pure and mixed strategies

Group rationality vs. individual rationality

Example (Penny-matching)

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>T</td>
<td>-1</td>
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</tr>
</tbody>
</table>

Definition

In a *zero-sum* game, one player wins whatever the other loses.

- no need to record payoff to both players
- no room for cooperation
Example (Penny-matching)

\[
\begin{bmatrix}
H & T \\
H & 1 & -1 \\
T & -1 & 1
\end{bmatrix}
\]

Definition

In a game of *perfect information*, both players know all moves that have already occurred.

- Examples: checkers, chess, tic-tac-toe.
- Nonexamples: poker, penny-matching, prisoner’s dilemma.
Definition (Group rationality)
An outcome is *Pareto optimal* if there is no other outcome which would give higher payoffs to everyone.

Definition (Individual rationality)
Pure strategy ① *dominates* pure strategy ② iff playing ① guarantees the payoff is at least as good as playing ②.

\[
\begin{array}{cccc}
\text{①}_C & \text{②}_C & \text{③}_C & \text{④}_C \\
\hline
\text{①}_R & 3 & -6 & 2 & -4 \\
\text{②}_R & 2 & 1 & 0 & 1 \\
\text{③}_R & -4 & 3 & -5 & 4 \\
\end{array}
\]

Pareto principle vs. dominance principle.
Definition (Equilibrium solution)

An *equilibrium solution* in a zero-sum game is a saddle point; an outcome which is not less than any other in its column, and not greater than any other in its row.

\[
\begin{array}{cccc}
1_C & 2_C & 3_C & 4_C \\
1_R & 12 & -1 & 1 & 0 \\
2_R & 5 & 1 & 7 & -20 \\
3_R & 3 & 2 & 4 & 3 \\
4_R & -16 & 0 & 0 & 16 \\
\end{array}
\]
Definition (Equilibrium solution)

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\begin{array}{c|c|c|c}
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\hline
5 & 1 & 7 & -20 \\
\hline
3 & 2 & 4 & 3 \\
\hline
-16 & 0 & 0 & 16 \\
\end{array}
\]
The Prisoner’s Dilemma (PD) is two-player game. Non-zero-sum (so cooperation may be worthwhile), but does not have perfect information (so cooperation is tricky).

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<td>$(0,-10)$</td>
<td>$(-6,-6)$</td>
</tr>
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</table>

Use movement diagram to find equilibrium outcome:

$$(-1, -1) \rightarrow (-10, 0)$$

$$\downarrow$$

$$(-10, 0) \downarrow$$

$$\downarrow$$

$$(-6, -6)$$
The Prisoner’s Dilemma (PD) is a two-player game. Non-zero-sum (so cooperation may be worthwhile), but does not have perfect information (so cooperation is tricky).

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Key features:

1. there is a unique (stable) equilibrium outcome
2. this outcome is not Pareto optimal
3. conflict between group rationality vs. individual rationality

(1950) PD is devised by Dresher and Flood of RAND to provide an example of 1 & 2.
Mathematical modeling of conflict scenarios

The Prisoner's Dilemma

Evolutionarily Stable Strategies

Multiplayer Prisoner's Dilemma

Iterated Prisoner's Dilemma

Generalized (Symmetric) PD

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<td>(S,T)</td>
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<td>(T,S)</td>
<td>(U,U)</td>
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</table>

with $T > R > U > S$ and $R \geq (S + T)/2$.

- $R$: reward for cooperation
- $S$: sucker payoff
- $T$: temptation payoff
- $U$: uncooperative payoff
The Prisoner’s Dilemma: real-world examples

Bidding war:
①: do not cut prices
②: cut prices.

- If the other merchant doesn’t cut prices, you should cut prices to gain more customers.
- If the other merchant does cut prices, you should cut prices to avoid losing customers.

Solution to PD: government intervention?
The Prisoner’s Dilemma: real-world examples

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Solution to PD: government intervention?

Practical consequence: Wal-Mart.
The Prisoner’s Dilemma: real-world examples

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Solution to PD: government intervention?
- Practical consequence: Wal-Mart.
- Practical consequence: Unions.
  (labour market: people offer to work for less).
The Prisoner’s Dilemma: real-world examples

- Instance: Criminals in jail — the original PD.
- Solution: baseball bats.
The Prisoner’s Dilemma: real-world examples

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- Solution: baseball bats.

The mafia: don’t deal with the DA.
①: remain silent and do your time.
②: sing like a birdie.
The tragedy of the commons


- each of 10 farmers in the community uses a common pasture
- pasture supports $\sim 1000$ cows, so each farmer raises 100
- each farmer considers raising a few extra cattle
  - cost: additional cow helps overgraze commons
    (but this cost is spread out and small to just one farmer)
  - benefit: a lot of money/food
Other Multiplayer PD examples

Pollution
①: dump within EPA guidelines.
②: dump a little extra.
Profits from extra dumping: +2
Environmental pollution: -4

\[ T > R > U > S \text{ and } R \geq (S + T)/2. \]
- \( R \): 0; status quo.
- \( S \): -4; not saving money, competitors are doing better, maybe enviro damage too!
- \( T \): +2; saving money & nobody noticed.
- \( U \): -2; saving money, but environmental damage (legal penalty?).
The Prisoner’s Dilemma: real-world examples

Cold War “Deterrence”: Mutual Assured Destruction (MAD)

“To continue to deter in an era of strategic nuclear equivalence, it is necessary to have nuclear (as well as conventional) forces such that in considering aggression against our interests any adversary would recognize that **no plausible outcome would represent a victory** or any plausible definition of victory.”

President Carter (1980)

Presidential Directive 59
Nuclear Weapons Employment Policy
The Prisoner’s Dilemma: real-world examples

Cold War “Deterrence”: Mutual Assured Destruction (MAD)

1: disarm.
2: keep nukes.

- **R**: +2; world becomes safer.
- **S**: -4; they have nukes and you don’t.
- **T**: +4; only you have nukes (world is safe and you’re in charge).
- **U**: -2; they have nukes, but you can still employ deterrence.

Equilibrium outcome: world teeters on brink of nuclear annihilation.

Solution to PD: nonproliferation disarmament negotiations. By shifting to *Iterated PD*, build trust, begin cooperation.
Iterated Prisoner’s Dilemma

Rose and Colin play this game many times:

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with $T > R > U > S$ and $R \geq (S + T)/2$.
Non-Markov: each player remembers what the other does!
Iterated Prisoner’s Dilemma

Rose and Colin play this game *many times*:

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with \(T > R > U > S\) and \(R \geq (S + T)/2\).

Non-Markov: each player remembers what the other does!

Suppose Rose & Colin are to play 10 times. There is no point cooperating on the 10\(^{th}\) play, so use \(\circ 2\). Reverse dominoes!
Assume another play will occur with probability $0 < p < 1$. First play occurs with probability 1, second with probability $p$, etc. (Big) Assumption: opponent is decent, but not forgiving.

If always choose $\textcircled{1}$, expected payoff is

$$R + pR + p^2R + p^3R + \cdots = \frac{R}{1 - p}.$$
Assume another play will occur with probability $0 < p < 1$. First play occurs with probability 1, second with probability $p$, etc. (Big) Assumption: opponent is decent, but not forgiving.

If always choose ①, expected payoff is

$$R + pR + p^2 R + p^3 R + \cdots = \frac{R}{1 - p}.$$  

If I choose ② in the $k^{th}$ play, expected payoff is

$$R + pR + \cdots + p^{k-1} R + p^k T + p^{k+1} U + p^{k+2} U + \cdots.$$  

So should never pick ② if the first quantity is larger, i.e., if

$$p > \frac{T - R}{T - U}.$$
Axelrod’s 1984 tournament: invite a bunch of people to submit programs to play the game.

14 contestants, some of considerable complexity. Each program plays the game against the others.
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14 contestants, some of considerable complexity. Each program plays the game against the others.

Winner: TIT-FOR-TAT, a four line program by Anatol Rapoport.

TIT-FOR-TAT strategy:

1. Start by playing $\text{tit}_1$ on the first round.
2. Choose whatever your opponent chose last time.
Axelrod’s 1986(?) tournament: Round Two, a couple of years later.

62 contestants, including many programs intricately designed to beat TIT-FOR-TAT (TFT).

Rapaport entered TFT again, unchanged.
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Rapaport entered TFT again, unchanged.

**Winner: TIT-FOR-TAT.**

The TFT-killers did well against TFT, but not against each other.

Final analysis: to do well, a strategy should be

1. Nice. Start by cooperating, never be the first to betray.
3. Forgiving. Having punished betrayal, should be willing to cooperate again.
4. Clear. Should be consistent and easy to predict.
Consider a game where the players are two birds, competing for some 50pt resource (food, mate, territory, etc).

Possible behaviors:
1. =dove: engage in symbolic conflict but don’t fight seriously.
2. =hawk: fight to kill, claws out.

dove vs. hawk: dove leaves immediately, hawk gets resource
dove vs. dove: one gets resource, both waste time & energy (-10)
hawk vs. hawk: one gets resource, other sustains injury (-100)

\[
\begin{array}{c|cc}
\text{dove}_R & \text{dove}_C & \text{hawk}_C \\
\hline
(15,15) & (0,50) \\
(50,0) & (-25,-25) \\
\end{array}
\]
Mathematical modeling of conflict scenarios
The Prisoner’s Dilemma
Evolutionarily Stable Strategies
Hawks vs. doves
Other possible strategies
Dating and mating

This game is not PD: $S = 0 \not< -25 = U$.
Movement diagram:

$$
\begin{array}{cc}
\text{dove}_R & \text{hawk}_R \\
(15,15) & (50,0) \\
(0,50) & (-25,-25)
\end{array}
$$

Erin Pearse
The Prisoner's Dilemma
Important: birds do not calculate their strategy rationally. They are *birds*, after all.

Evolutionary pressure (aka natural selection) replaces rationality.

Ansatz: bird is “hardwired” to favour some strategy. What is the distribution of behaviors (strategies) over time?
Important: birds do not calculate their strategy rationally. They are *birds*, after all.

Evolutionary pressure (aka natural selection) replaces rationality.

Ansatz: bird is “hardwired” to favour some strategy. What is the distribution of behaviors (strategies) over time?

**Definition (Evolutionarily Stable Strategy)**

Suppose that everyone in a population is playing strategy $S$. Let $T$ be any other strategy. Then $S$ is an *evolutionarily stable strategy* (ESS) if and only if the expected payoff for a newcomer playing $S$ is at least as great as a newcomer playing $T$.

Actually, think of “the newcomer” as a small group of potential invaders.
Suppose almost the entire population consists of doves. A dove can expect an average of 15 per interaction. A (rare) hawk can expect 50 per interaction. Consequence: a small population of hawks can invade! The pure strategy **dove** is not evolutionarily stable.
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Suppose almost the entire population consists of hawks. A hawk can expect an average of -25 per interaction. A (rare) dove can expect 0. Consequence: a small population of doves can invade! The pure strategy **hawk** is not evolutionarily stable.

An ESS for this game must be a *mixed* strategy. Equiv, an evolutionarily stable population must contain hawks *and* doves.

What is the ESS?
Computing an ESS

\[
\begin{array}{c|cc}
\text{dove} & \text{dove} & \text{hawk} \\
\text{dove}_R x & 15 & 0 \\
\text{hawk}_R (1 - x) & 50 & -25 \\
\end{array}
\]

\[15x + 50(1 - x) = (-25)(1 - x) \implies x = \frac{5}{12}.\]
### Computing an ESS

<table>
<thead>
<tr>
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<th>Dove</th>
<th>Hawk</th>
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<td>0</td>
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<tr>
<td>Hawk</td>
<td>50</td>
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\[
15x + 50(1 - x) = (-25)(1 - x) \implies x = \frac{5}{12}.
\]

So an ESS is a population with \(\frac{5}{12}\) doves and \(\frac{7}{12}\) hawks. Equivalently, a population where each individual acts like a dove \(\frac{5}{12}\) of the time and a hawk the other \(\frac{7}{12}\).

A mutant with different mixed strategy will not prosper here, even though the expected payoff is 6.25. This is not Pareto optimal! \((6.25 < 15)\)

\[
\frac{5}{12}(50) + \frac{7}{12}(-25) = 6.25 = \frac{5}{12}(15) + \frac{7}{12}(0).
\]
A dove-beating strategy

ESS drawback: must identify *all feasible strategies*.

**Bully**: In any contest, begin fighting like a hawk but if the opponent fights like a hawk, then flee!

Payoffs to Rose:

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<td>50</td>
<td>0</td>
<td>25</td>
</tr>
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</table>

Bully dominates dove, so doves will die out.

The resulting (unique) ESS is $\frac{1}{2}$ hawks, $\frac{1}{2}$ bullies.
A bully-beating strategy

**Retaliator**: In any contest, begin like a dove. However, if opponent attacks, fight back relentlessly!

Retaliators act like hawks against hawks and like doves against doves — the worst of both worlds!

Payoffs to Rose:

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The pure strategy **retaliator** is an ESS!
So is *any* mixture of retaliators and doves with doves ≤ 30%.

HW #1: Prove that bullies can invade when doves > 30%.
In a population of retaliators, there is no real fighting, only posturing & threatening.

Real-life examples:
1. bighorn sheep
2. venomous snakes
3. crabs
4. birds of all kinds

Retaliator has similar flavour to TFT, shares many qualities.
The asymmetric mating game

A population in which it is advantageous to the female to have the male stay and help raise the offspring.

coy: insist on a long arduous courtship before mating
fast: mate with anyone

Male options: protect genes or distribute them widely.
faithful: go through courtship, help raise offspring
philandering: no courtship, depart after mating

Payoffs: babies = +15 each, courtship = −3 to each, raising babies = −20 (split equally, or fall entirely on female).

<table>
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<tr>
<td>coy</td>
<td>(2,2)</td>
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The asymmetric mating game

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(2, 2) ← (0, 0)

(5, 5) → (−5, 15)

Female ESS: \(\frac{5}{6}\) coy, \(\frac{1}{6}\) fast.

Male ESS: \(\frac{5}{8}\) faithful, \(\frac{3}{8}\) philandering.

Expected payoff: \((1 \frac{1}{4}, 2 \frac{1}{2}) < (5, 5)\), not Pareto optimal.
For Further Reading . . .

Philip D. Straffin

*Game Theory and Strategy.*
Rose and Colin are dating

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<td>(4,1)</td>
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Rose always likes to go to the movies. Colin would rather watch the game. Each is miserable without the other.

Three equilibrium pairs. Pure strategies: (1, 4) and (4, 1). Mixed strategies: $R = (\frac{1}{5}, \frac{4}{5})$ vs $C = (\frac{4}{5}, \frac{1}{5})$, with payoffs $(\frac{4}{5}, \frac{4}{5})$. 
Rose and Colin are dating

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\[
P_R(x, y) = 1xy + 0x(1 - y) + 0y(1 - x) + 4(1 - x)(1 - y) \\
= (5y - 4)x - 4y + 4
\]
Rose and Colin are dating

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\[ \begin{align*}
P_R(x, y) &= (5y - 4)x - 4y + 4, \\
P_C(x, y) &= (5x - 1)y - x + 1
\end{align*} \]
Rose and Colin are dating

The Prisoner’s Dilemma

\[
\begin{array}{c|c|c}
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\hline
(1,4) & (0,0) \\
(0,0) & (4,1) \\
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\[
P_R(x, y) = (5y - 4)x - 4y + 4, \quad P_C(x, y) = (5x - 1)y - x + 1
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Rose and Colin are dating

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<td>(4, 1)</td>
</tr>
</tbody>
</table>

With cooperation, can get any payoff in the convex hull.