Self-similar fractals as boundaries of networks

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4th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals

AMS Eastern Sectional Meeting, Fall 2011
Sep. 12, 2011
Previously work relating self-similar fractals to boundaries

[DS] Denker and Sato construct a Markov chain for which the Sierpinski gasket is (homeomorphic to) the Martin boundary.

[Kai] Kaimanovich outlines a program for understanding the Sierpinski gasket as a Gromov boundary, by constructing an associated hyperbolic graph.

[Kig2] Kigami develops the theory of resistance analysis on trees, obtains results on jump process on the “boundary” Cantor set.

[LW] Lau and Wang extend Kaimanovich’s construction to self-similar sets satisfying the open set condition.

[JLW] Ju, Lau, and Wang extend Denker & Sato’s work to a certain class of pcf self-similar fractals.
Definition (Networks \((G, c)\))

A network \((G, c)\) is a connected simple graph.

\(x \in G\) means \(x\) is a vertex.

The edges are determined by a weight function called \textit{conductance}:

\(x \sim y\) iff \(0 < c_{xy} < \infty\).

\(x \not\sim y\) iff \(c_{xy} = 0\).

\(c\) is symmetric: \(c_{xy} = c_{yx}\) for all \(x, y \in G\).

Total conductance at a vertex must be finite:

\[ c(x) := \sum_{y \sim x} c_{xy} < \infty, \text{ for all } x \in G. \]
Network energy and Laplacian

Definition ((Dirichlet) energy of a function $u : G \to \mathbb{R}$)

$$\mathcal{E}(u) := \frac{1}{2} \sum_{x, y \in G} c_{xy} |u(x) - u(y)|^2, \quad \text{dom } \mathcal{E} = \{u : \mathcal{E}(u) < \infty\}.$$  

$c_{xy} = 0$ unless $x \sim y$, so “$\frac{1}{2}$” means each edge is counted once.

Note: $\text{Ker } \mathcal{E} = \{\text{constant functions}\}$.

For $f : \mathbb{R}^n \to \mathbb{R}$, the continuous analogue is $\mathcal{E}(f) := \int_U |\nabla f|^2 \, dV$. 
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Definition (Energy form $\mathcal{E}$ on functions $u, v \in \text{dom } \mathcal{E}$)

$$\mathcal{E}(u, v) := \frac{1}{2} \sum_{x,y \in G} c_{xy} (u(x) - u(y))(v(x) - v(y))$$
Network energy and Laplacian

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$$

$c_{xy} = 0$ unless $x \sim y$, so “$\frac{1}{2}$” means each edge is counted once.

Definition (Laplace operator)

The Laplacian $\Delta$ is a symmetric operator acting on $u : G \to \mathbb{R}$ by

$$
(\Delta u)(x) := \sum_{y \sim x} c_{xy}(u(x) - u(y))
$$

$u$ is harmonic iff $\Delta u = 0$. 

The IFS \( \{ \Phi_0, \Phi_1, \ldots, \Phi_J \} \)

\( \Phi_j \) is a rigid motion of \( \mathbb{R}^d \) + homothety with scaling factor \( r_j \in (0, 1) \).

\( \tilde{V}_0 = \{ q_0, q_1, \ldots, q_J \} \) are the fixed points of \( \Phi_j \) and

\[
\Phi (A) := \bigcup_{j=0}^J \Phi_j (A)
\]

\( \mathcal{F} \) is the unique nonempty compact subset with \( \Phi(\mathcal{F}) = \mathcal{F} \).

Assume \( \mathcal{F} \) carries a regular harmonic structure.

(Need the self-similar energy form on \( \mathcal{F} \) to have renomalization factors are < 1.)
The IFS $\{\Phi_0, \Phi_1, \ldots, \Phi_J\}$

$\Phi_j$ is a rigid motion of $\mathbb{R}^d$ + homothety with scaling factor $r_j \in (0, 1)$.

$\tilde{V}_0 = \{q_0, q_1, \ldots, q_J\}$ are the fixed points of $\Phi_j$ and

$$\Phi(A) := \bigcup_{j=0}^{J} \Phi_j(A)$$

$\mathcal{F}$ is the unique nonempty compact subset with $\Phi(\mathcal{F}) = \mathcal{F}$.

Assume $\mathcal{F}$ carries a regular harmonic structure.

$\mathcal{W}_m := \{0, 1, \ldots, J\}^m$ is the set of words $w = w_1w_2 \ldots w_m$.

$$\Phi_w(x) := \Phi_{w_1} \circ \Phi_{w_2} \circ \ldots \circ \Phi_{w_m}(x)$$

The set $\Phi_w(\mathcal{F})$ is called a $m$-cell. Here $|w| = m$. 
Discrete approximants of $\mathcal{F}$

$\tilde{V}_0 = \{q_0, q_1, \ldots, q_J\}$ are the fixed points of $\Phi_j$.

$\tilde{\Gamma}_0$ is the complete network on $\tilde{V}_0$. For now, $c_{xy} = 1$. Define

$$\tilde{\Gamma}_k := \Phi(\tilde{\Gamma}_{k-1}) = \Phi^k(\tilde{\Gamma}_0),$$

so $\tilde{\Gamma}_k$ has vertices $\tilde{V}_k = \Phi^k(\tilde{V}_0)$ and edges $c_{\Phi_j(x)\Phi_j(y)} = c_{xy}$.

$\tilde{V}$ refers to vertices of $\mathcal{F}$, $V$ refers to vertices of $\mathcal{N}_\mathcal{F}$. 

![Diagram of discrete approximants](image)
The Sierpinski gasket $SG$ and Sierpinski network $N_{SG}$
Discrete approximants of $\mathcal{F}$

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$\tilde{V}$ refers to vertices of $\mathcal{F}$, $V$ refers to vertices of $\mathcal{N}_\mathcal{F}$.

The vertex set $V$ of $\mathcal{N}_\mathcal{F}$ is a subset of $\mathcal{F} \times \mathbb{Z}_+$:

$$V_k := \tilde{V}_k \times \{k\} \quad \text{and} \quad \Gamma_k := \tilde{\Gamma}_k \times \{k\}.$$

Then $V := \bigcup_{k=0}^{\infty} V_k$ is the vertex set of $\mathcal{N}_\mathcal{F}$. 
The edges of $\mathcal{N}_F$

Let $E_k$ denote the set of edges of the graph $\Gamma_k$ previously defined.

The set of *horizontal edges* of $\mathcal{N}_F$ is $E = \bigcup_{k=1}^{\infty} E_k$. 

![Diagram showing the edges of $\mathcal{N}_F$](image-url)
The edges of $\mathcal{N}_F$

Let $E_k$ denote the set of edges of the graph $\Gamma_k$ previously defined.

The set of *horizontal edges* of $\mathcal{N}_F$ is $E = \bigcup_{k=1}^{\infty} E_k$.

The set of *vertical edges* of $\mathcal{N}_F$ is $F_k$:

$$
(\Phi_{wj}(q_i), k) \\
| \\
(\Phi_w(q_i), k - 1)
$$

Here, $(\Phi_w(q_i), k - 1) \in V_{k-1}$ and $i, j \in \{1, 2, \ldots, J\}$

The set of all vertical edges is $F = \bigcup_{k=1}^{\infty} F_k$. 
The Sierpinski gasket $SG$ and Sierpinski network $N_{SG}$
Compactifications

If $\Omega \subseteq \mathbb{C}$ or $\mathbb{R}^d$, it is clear what $\partial\Omega$ is.

If $\Omega$ is the state space of a random walk, $\partial\Omega$ describes limits.

To obtain a boundary: use a family of functions $\mathcal{U}$ to obtain a compactification $\hat{\Omega}$. Then $\partial\Omega = \hat{\Omega} \setminus \Omega$. 
Compactifications

If $\Omega \subseteq \mathbb{C}$ or $\mathbb{R}^d$, it is clear what $\partial \Omega$ is.

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Let $\mathcal{U}$ be a family of bounded functions. Then equivalently:

(i) Take $\hat{\Omega} = \Omega_\infty / \sim$, where $\Omega_\infty$ is the space of infinite paths in $\Omega$ and 
\[ (x_n) \sim (y_n) \text{ iff } \lim f(x_n) = \lim f(y_n) \text{ for every } f \in \mathcal{U}. \]

(ii) Take $\hat{\Omega}$ to be the smallest space to which each $f \in \mathcal{U}$ can be (continuously) extended.

(iii) Take a metric completion, using 
\[ d(x, y) = \sum_{f \in \mathcal{U}} w_f |f(x) - f(y)|, \text{ for weights } w_f > 0. \]
Compactifications

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For the **Martin boundary**, use $\mathcal{U} = \{ \mathbb{1}_k(x, \cdot) : x \in \Omega \}$, where

$$\mathbb{1}_k(x, y) = \frac{g(x, y)}{g(o, y)}.$$ 

$$g(x, y) = \mathbb{E}[\# \{X_n = y\} | X_0 = x] = \sum_{n=0}^{\infty} p_n(x, y).$$
Compactifications

If $\Omega \subseteq \mathbb{C}$ or $\mathbb{R}^d$, it is clear what $\partial \Omega$ is.

If $\Omega$ is the state space of a random walk, $\partial \Omega$ describes limits.

To obtain a boundary: use a family of functions $\mathcal{U}$ to obtain a compactification $\hat{\Omega}$. Then $\partial \Omega = \hat{\Omega} \setminus \Omega$.

**Theorem:** For every harmonic and bounded function $h$ on $\Omega$, there is a Borel measure $\nu^h$ on $\partial \Omega$ such that

$$h(x) = \int_{\partial \Omega} \mathbb{1}_k(x, \cdot) \, d\nu^h, \quad \text{for every } x \in \partial \Omega.$$ 

Next: we replace the Martin kernels with a family $\{u_x\}_{x \in G}$. 
Harmonically generated functions

Define a function $u$ on $\mathcal{N}_F$ as follows:

1. Specify the values on $V_0$.
2. Let $A_{w_1}, \ldots, A_{w_J}$ be the harmonic extension matrices.
3. Let $\Psi_j(x, z) = (\Phi_j(x), z + 1)$.
4. Define $u$ on the remainder of $\mathcal{N}_F$ via $u|_{\Psi_w V_0} = A_w u|_{V_0}$.

Lemma. The harmonically generated function $u$ has finite energy.
Harmonically generated functions

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Lemma. The harmonically generated function $u$ has finite energy.

$$E(u) = \sum_{k=0}^{\infty} E_{E_k}(u) + \sum_{k=1}^{\infty} E_{F_k}(u) \leq 2 \sum_{k=0}^{\infty} r^k E_{E_0}(u) = \frac{E_{E_0}(u)}{1 - r} < \infty.$$ 

Note: $r < 1$ follows from regularity assumption.
Harmonically generated functions

Define a function $u$ on $\mathcal{N}_F$ as follows:

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2. Let $A_{w_1}, \ldots, A_{w_J}$ be the harmonic extension matrices.
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4. Define $u$ on the remainder of $\mathcal{N}_F$ via $u|_{\Psi_w V_0} = A_w u|_{V_0}$.

Lemma. The harmonically generated function $u$ has finite energy.

For $x \in V_0$, let $u_x$ denote the harmonically generated function initiated with

$$u_x(y) = \begin{cases} 1, & y = x \\ 0, & y \in V_0 \setminus \{x\}. \end{cases}$$
Localized harmonically generated functions

For \( x = (\xi, m) \in \mathcal{N}_\mathcal{F} \), let \( K(\xi, m) \) be the union of the \( m \)-cells of \( \mathcal{F} \) containing \( \xi \).

Repeat the construction of the harmonically generated function for \( u \) on the vertices below these cells, extend by 0 elsewhere:

\[
    u_x(z) := \begin{cases} 
    u(\Psi_{w(q)}^{-1}(z)), & z \in \Psi_{w(q)}(\mathcal{N}_\mathcal{F}) \text{ for some } q \in V_0, \\
    0, & \text{else.}
    \end{cases}
\]

\[
    \lim_{n \to \infty} u_x(\xi, n) = 1, \quad \lim_{n \to \infty} u_x(\zeta, n) = 0, \text{ for } \zeta \notin K(\xi, m). \quad y = (\xi, m + 1).
\]
Localized harmonically generated functions

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Repeat the construction of the harmonically generated function for \( u \) on the vertices below these cells, extend by 0 elsewhere:

\[
 u_x(z) := \begin{cases} 
  u(\Psi^{-1}_{w(q)}(z)), & z \in \Psi_{w(q)}(\mathcal{N}_\mathcal{F}) \text{ for some } q \in V_0, \\
  0, & \text{else}.
\end{cases}
\]

Use the family \( \mathcal{U} := \{u_x\}_{x \in \mathcal{N}_\mathcal{F}} \) to compactify \( \mathcal{N}_\mathcal{F} \).
Path space in $\mathcal{N}_\mathcal{F}$

A path in $\mathcal{N}_\mathcal{F}$ is a sequence of adjacent vertices 

$$(x_n)_{n=1}^\infty, \quad x_n = (\xi_n, k_n) \in \mathcal{N}_\mathcal{F}, \quad \xi_n \in \mathcal{F}, k_n \in \mathbb{Z}_+.$$ 

The transition probabilities $p(x, y) := \frac{c_{xy}}{c(x)}$ induce a natural measures on the trajectory space $\Omega = \{\text{paths in } \mathcal{N}_\mathcal{F}\}$. 
Path space in $\mathcal{N}_F$

A path in $\mathcal{N}_F$ is a sequence of adjacent vertices

$$(x_n)^\infty_{n=1}, \quad x_n = (\xi_n, k_n) \in \mathcal{N}_F, \quad \xi_n \in F, k_n \in \mathbb{Z}_+.$$

The transition probabilities $p(x, y) := c_{xy}/c(x)$ induce a natural measures on the trajectory space $\Omega = \{\text{paths in } \mathcal{N}_F\}$.

Define an equivalence relation on $\Omega$ by

$$(x_n)^\infty_{n=1} \sim (y_n)^\infty_{n=1} \iff \lim_{n \to \infty} u_z(x_n) = \lim_{n \to \infty} u_z(y_n), \text{ for all } z \in G,$$

where $u_z$ are harmonically generated.

The boundary of $\mathcal{N}_F$ is $\mathcal{B}_F := \Omega/\sim$,

and $\mathcal{\hat{N}}_F = \mathcal{N}_F \cup \mathcal{B}_F$ is a compactification of $\mathcal{N}_F$. 
The boundary $B_\mathcal{F}$ of the network $\mathcal{N}_\mathcal{F}$

Compactification of $\mathcal{N}_\mathcal{F}$

Path space in $\mathcal{N}_\mathcal{F}$

A path in $\mathcal{N}_\mathcal{F}$ is a sequence of adjacent vertices

$$(x_n)_{n=1}^\infty, \quad x_n = (\xi_n, k_n) \in \mathcal{N}_\mathcal{F}, \quad \xi_n \in \mathcal{F}, k_n \in \mathbb{Z}_+.$$ 

The transition probabilities $p(x, y) := c_{xy}/c(x)$ induce a natural measures on the trajectory space $\Omega = \{\text{paths in } \mathcal{N}_\mathcal{F}\}$.

$\mathbb{P}_x$ is the corresponding measure on paths starting at $x \in \mathcal{N}_\mathcal{F}$.

**Lemma.** For any $x \in \mathcal{N}_\mathcal{F}$, the random walk started at $x$ is transient.
The boundary \( B_\mathcal{F} \) of the network \( \mathcal{N}_\mathcal{F} \)

**Theorem (Convergence to the boundary).**

There is a \( \mathcal{B}_\mathcal{F} \)-valued r.v. \( X_\infty \) such that for every \( x \in \mathcal{N}_\mathcal{F} \),

\[
\lim_{n \to \infty} X_n = X_\infty, \quad \mathbb{P}_x \text{-almost surely,}
\]

in the topology of \( \widehat{\mathcal{N}_\mathcal{F}} \).
Theorem (Convergence to the boundary).
There is a $\mathcal{B}_\mathcal{F}$-valued r.v. $X_\infty$ such that for every $x \in \mathcal{N}_\mathcal{F}$,
\[
\lim_{n \to \infty} X_n = X_\infty, \quad \mathbb{P}_x\text{-almost surely},
\]
in the topology of $\widehat{\mathcal{N}_\mathcal{F}}$.

In other words, if
\[
\Omega_\infty := \{(x_n)_{n=1}^\infty \in \Omega : x_n \to x_\infty \in \mathcal{B}_\mathcal{F} \text{ in the topology of } \widehat{\mathcal{N}_\mathcal{F}}\},
\]
then:

1. $\Omega_\infty$ is measurable w.r. to $\mathcal{A}$, the Borel $\sigma$-algebra of $\widehat{\mathcal{N}_\mathcal{F}}$.
2. For every $x \in \mathcal{N}_\mathcal{F}$, one has $\mathbb{P}_x(\Omega_\infty) = 1$.
3. For each $x \in \mathcal{N}_\mathcal{F}$, the function $X_\infty : \Omega \to \mathcal{B}_\mathcal{F}$ defined by
\[
X_\infty(\omega) = x_\infty, \quad \text{for all } \omega = (x_n) \in \Omega_\infty,
\]
is measurable with respect to $\mathcal{A}$. 
Theorem (Convergence to the boundary).
There is a $\mathcal{B}_F$-valued r.v. $X_\infty$ such that for every $x \in \mathcal{N}_F$,
\[ \lim_{n \to \infty} X_n = X_\infty, \quad \mathbb{P}_x\text{-almost surely}, \]
in the topology of $\hat{\mathcal{N}}_F$.

The main idea for the second part:
- For every $x \in \mathcal{N}_F$, one has $\mathbb{P}_x(\Omega_\infty) = 1$.

For Martin boundary, convergence to the boundary comes from
\[ \mathbb{I}k(x, y) \text{ superharmonic} \implies \mathbb{I}k(X_n, y) \text{ supermartingale}. \]

For this construction, convergence to the boundary comes from
\[ u_y(x) \text{ finite-energy} \implies \lim_{n \to \infty} u_y(X_n) \text{ converges } \mathbb{P}_x\text{-a.e.}, \]
by a key result in [ALP].
Theorem (Identification of the boundary).
The network boundary $B_F$ is (homeomorphic to) the attractor $\mathcal{F}$.

Idea of proof:

- Extend the $u_x$ functions to $\widehat{\mathcal{N}}_F$ by continuity.
- These separate points of $\mathcal{F}$, so can be used to establish a bijection $f : \mathcal{F} \to B_F$.
- For any point rational point $\xi \in \mathcal{F}$, the family $\{K(\xi, m)\}_{m \geq m_0}$ forms a neighbourhood basis at $\xi$.
- For $\xi \in \mathcal{F}$ and $x \in V_m$, one has $u_x(f(\xi)) = \psi_x^{(m)}(\xi)$. Here $\psi_x^{(m)}$ is the piecewise harmonic spline of [Str, Thm. 2.1.2].
- For subbasic open sets $U_{x,x_\infty,\varepsilon} := \{y_\infty \in B_F : |u_x(x_\infty) - u_x(y_\infty)| < \varepsilon\}$, show $f^{-1}(U_{x,x_\infty,\varepsilon})$ is preimage of an open set under $\psi_x^{(m)}$.
- A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.
Trace of the energy form

Let $\tilde{\Omega} = \Omega \cup B$ and $\varphi \in L^\infty(B)$.

(Q): What is the energy of $\varphi$?
Trace of the energy form

Let $\tilde{\Omega} = \Omega \cup \mathcal{B}$ and $\varphi \in L^\infty(\mathcal{B})$.

(Q): What is the energy of $\varphi$?

(A): Take the harmonic extension of $\varphi$ to $\tilde{\Omega}$:

$$ h_\varphi(x) = \mathbb{E}_x(\varphi(X_\infty)) = \int_{\mathcal{B}} \varphi \, d\nu_x, \quad \text{for all } x \in \Omega. $$

Now $h_\varphi$ is harmonic on $\Omega$, so the energy-minimizing extension:

$$ \Delta h_\varphi(x) = 0, \text{ for } x \in \Omega \quad \text{and} \quad h_\varphi(x) = \varphi(x), \text{ for } x \in \mathcal{B}. $$
Trace of the energy form

Let $\tilde{\Omega} = \Omega \cup B$ and $\varphi \in L^\infty(B)$.

(Q): What is the energy of $\varphi$?

(A): Take the harmonic extension of $\varphi$ to $\tilde{\Omega}$:

$$h_\varphi(x) = \mathbb{E}_x(\varphi(X_\infty)) = \int_B \varphi \, d\nu_x, \quad \text{for all } x \in \Omega.$$ 

Now $h_\varphi$ is harmonic on $\Omega$, so the energy-minimizing extension:

$$\Delta h_\varphi(x) = 0, \quad \text{for } x \in \Omega \quad \text{and} \quad h_\varphi(x) = \varphi(x), \quad \text{for } x \in B.$$

Define the trace energy form:

$$\mathcal{E}_B(\varphi) := \mathcal{E}(h_\varphi), \quad \text{dom } \mathcal{E}_B = \{ \varphi \in L^\infty(B) : h_\varphi \in \text{dom } \mathcal{E} \}.$$
Self-similar fractals as boundaries of networks

The boundary $\mathcal{B}_F$ of the network $\mathcal{N}_F$

Harmonic extension

Work in progress:

**Theorem (Harmonic extension).**

Let $\varphi : F \rightarrow \mathbb{R}$ satisfy $\mathcal{E}_F(\varphi) < \infty$. Then for any $x \in \mathcal{N}_F$, there is a measure $\mu_x$ on $F$ such that

$$h_\varphi(x) = \int_F \varphi(\xi) \, d\mu_x(\xi)$$

defines a harmonic function of finite energy on $\overline{\mathcal{N}_F}$.

- What conditions ensure a harmonic extension $h_\varphi$?
- What conditions on $h$ ensure a harmonic representation in terms of some $\varphi$?
- Can one establish a relation between $\mathcal{E}(h_\varphi)$ and $\mathcal{E}_F(\varphi)$?

Here $\mathcal{E}_F$ is the energy on $F$ as defined in [Kig1] (see also [Str]). Recent related results: [WL, LN].
Alano Ancona, Russell Lyons, and Yuval Peres. Crossing estimates and convergence of Dirichlet functions along random walk and diffusion paths. 


Self-similar fractals as boundaries of networks

The boundary $B_F$ of the network $N_F$

Harmonic extension

Jun Kigami.  

Jun Kigami.  
Dirichlet forms and associated heat kernels on the Cantor set induced by random walks on trees.  

Ka-Sing Lau and Sze-Man Ngai.  
Martin boundary and exit space on the Sierpinski gasket and other fractals.  

Ka-Sing Lau and Xiang-Yang Wang.  
Self-similar sets as hyperbolic boundaries.  
Self-similar fractals as boundaries of networks

- The boundary $B_F$ of the network $N_F$

- Harmonic extension

Robert S. Strichartz.  
*Differential equations on fractals.*
A tutorial.

Ting-Kam Wong and Ka-Sing Lau.  
Random walks and induced Dirichlet forms on self-similar sets.  