Student Seminar: 9/23/2011.

Book: Algebraic Geometry by
Daniel Perrin.

Chapter I: Affine algebraic sets.
§ Affine alg. sets and Zariski top.

Througout
- k is a comm. field.
- n positive integer.
- $x = (x_1, \dots, x_n) \in k^n$.
- $P(X_1, \dots, X_n)$ is poly & $p(x) = p(x_1, \dots, x_n)$

Def/ Let $S \subseteq k[X_1, \dots, X_n]$.
The affine alg. set defined by S is
$$V(S) = \{ x \in k^n : p(x) = 0, \ \forall \ p \in S \}$$
i.e $x \in V(S)$ are the common zeros of all the polys in S.

. If S is finite i.e $S = \{P_1, \dots, P_r\}$
then we write $V(S) = V(P_1, \dots, P_r) = V(\{P_1, \dots, P_r\})$

Examples: the empty set ϕ and k^n are affine algebraic sets. b/c:
$$V(\{1\}) = \{ x \in k^n : p(x) = 0 \ \forall \ p \in \overset{\{1\}}{\overset{\shortparallel}{S}} \}$$
$$= \phi$$
$$V(\{0\}) = \{ x \in k^n : p(x) = 0 \ \forall \ p \in \{0\} \}$$
$$= k^n$$

$V(S) = \{ x \in k^n : p(x) = 0 \quad \forall \; p \in S \}$

Let $x \in V(S)$

$\Rightarrow \; p(x) = 0 \quad \forall \; p \in S$

then

$\forall \, \langle S \rangle \ni f = \sum_{i=1}^{r} a_i \, p_i$

$\Rightarrow \quad f(x) = \sum_{i=1}^{r} a_i \, p_i(x) = 0$ since $p_i(x) = 0 \; \forall \; i$.

$\Rightarrow \quad x \in V(\langle S \rangle)$

So $\quad V(S) \subset V(\langle S \rangle)$

Hence $\quad V(S) = V(\langle S \rangle)$.

(This important, b/c now instead of working with S we only works with ideals or the set of generator of the ideal).

3_ We know $k[x_1, \ldots, x_n]$ is Noetherian

$\Rightarrow$ any ideal I in $k[x_1, \ldots, x_n]$ is fi generated

$\Rightarrow \quad I = \langle p_1, \ldots, p_r \rangle$

$\Rightarrow \; V(I) = V(\langle p_1, \ldots, p_r \rangle) = V(p_1, \ldots, p_r)$

why $\cap$ not Union?? $\quad = V(p_1) \cap \cdots \cap V(p_r)$.

$x \in V(p_1, \ldots, p_r) = \{ x \in k^n : p_i(x) = 0 \quad \forall i \}$

i.e $\; p_1(x) = 0, \ldots, p_r(x) = 0$

$\Rightarrow \; x \in V(p_1), \ldots, x \in V(p_r)$ i.e $x \in V(p_1) \cap \cdots \cap V(p_r)$

Let $x \in V(p_1) \cap \cdots \cap V(p_r)$

$\Rightarrow \quad p_1(x) = 0, \ldots, p_r(x) = 0$

$\Rightarrow \quad x \in V(p_1, \ldots, p_r)$.

Hence $V(I) = V(p_1) \cap \cdots \cap V(p_r)$

$$x \in \bigcap_j V(S_j) \iff x \in V(S_j) \quad \forall j$$

$$\iff p_j(x) = 0 \quad \forall p_j \in S_j, \ \forall j$$

$$\iff p(x) = 0 \quad \forall p \in S_1$$
$$p(x) = 0 \quad \forall p \in S_2$$
$$\vdots$$

$$\iff p(x) = 0 \quad \forall p \in S_1 \cup S_2 \cup \cdots$$
i.e. $p(x) = 0 \quad \forall p \in \bigcup_j S_j$

$$\iff x \in V(\bigcup_j S_j) = \{x \in k^n : p(x) = 0 \ \forall p \in \bigcup_j S_j\}$$

$$\therefore \bigcap_j V(S_j) = V(\bigcup_j S_j)$$

7- A finite Union of alg. sets is an affine alg. set.

From (2) it suffices to show this for ideals.

Let I, J be two ideals in $k[x_1, \dots, x_n]$.

w.t.s. $V(I) \cup V(J) = V(IJ)$.

$$IJ = \{\sum_{i=1}^{r} x_i y_i : x_i \in I, y_i \in J\}$$

$IJ \subset I \cap J \subset I, J$.

then $V(IJ) \supset V(I)$
$V(IJ) \supset V(J)$ by (1)
$\Rightarrow V(I) \cup V(J) \subset V(IJ)$.

Now, let $x \in V(IJ)$ & let $x \notin V(I)$ $\Rightarrow$ $p(x) = 0 \quad \forall p \in IJ$

$$\Rightarrow p(x) = 0 \quad \forall p = \sum_{i=1}^{r} f_i g_i \in IJ.$$

Standard open sets are $D(f) = k^n - V(f)$
where $f \in k[x_1, \ldots, x_n]$ & $V(f)$ is the hypersurface.

§2 Ideal of an affine alg. set :

Def / Terminology :
let $V \subset k^n$ (wma. V is an affine alg. set)
then
define the set : $I(V) = \{ p \in k[x_1, \ldots, x_n] : p(x) = 0 \ \forall x \in V \}$

$I(V)$ is called the <u>ideal</u> of V.
(In words, $I(V)$ is the set of poly. that vanishes on V)

. To show $I(V)$ is indeed an ideal :
let $f, g \in I(V)$ i.e. $f(x) = 0 \quad \forall x \in V$
$g(x) = 0 \quad \forall x \in V$

then $(f+g)(x) = f(x) + g(x)$
$= 0 \quad \forall x \in V$

so $f+g \in I(V)$.

let $f \in I(V)$, $p \in k[x_1, \ldots, x_n]$
then $(fp)(x) = f(x) \cdot p(x)$
$= 0 \cdot p(x) \quad \forall x \in V$
$= 0 \quad \forall x \in V$

$\Rightarrow \quad fp \in I(V)$ Hence $I(V)$ is an ideal.

$I = \langle f_1, \dots, f_r \rangle \subset k[x_1, \dots, x_n]$

$I(V) = \{ f \in k[x_1, \dots, x_n] \, ; \, f(x) = 0 \quad \forall x \in V \}$

But $f_i(x) = 0 \quad \forall \; x \in V$ since $V = V(I)$

$\Rightarrow \; I \subset I(V)$.

$\Rightarrow \; V(I) \supset V(I(V))$ the map V is decreasing

Hence $V = V(I) = V(I(V))$.

3_ The map $\phi : k^n \longrightarrow k[x_1, \dots, x_n]$

I assumed V is affine $\quad V \longmapsto I(V)$

is injective.

prf: Let $V \neq V_1$ w.t.s. $I(V_1) \neq I(V_2)$

$\exists \; x \overset{\text{!!}}{=} (a_1, \dots, a_n) \in V$ & $x \notin V_1$

$x \in V = V(I(V))$ & $x \notin V_1(I(V_1))$

i.e $\exists \; p_1 \in I(V_1)$ s.t. $p_1(x) \neq 0$.

while $p(x) = 0 \;\; \forall \; p \in I(V)$.

$\Rightarrow \; I(V_1) \neq I(V)$.

Then if $V \subsetneq W \Rightarrow \exists$ a poly which vanishes at V and does not vanish on W.

prf: $I(V) \supset I(W)$ & since $V \neq W \Rightarrow I(V) \neq I(W)$.

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Proposition:

If k is infinite, then $I(k^n) = 0$.

prf:

Induction on n.

for $n=1$, $I(k) = \{ f \in k[x] : f(x) = 0 \ \forall x \in k \}$

$= \{0\}$ since $f = 1 \Rightarrow f(0) = 1 \neq 0$.

Assume true for all $r < n$.

for n, let $p \neq 0$ and non cte

then by Euclidean division:

$$p = a_r(X_1, \ldots, X_{n-1}) X_n^r + \cdots \qquad r \geq 1$$

$$\& \quad a_r \neq 0$$

$(X_1, \ldots, X_n)$ is the ideal generated by $X_1, \ldots, X_{n-1}$.

$\{ q_1(x) X_1 + \cdots q_n(x) X_n : q_i \in k[X_1, \ldots X_n] \}$

By induction step: $\exists \overset{(x_1, \ldots, x_n)}{\overset{\shortparallel}{x}} \in k^n$ s.t.

$$a_r(x_1, \ldots, x_n) \neq 0.$$

then $p(x_1, \ldots, x_n, X_n)$ has at most r roots and hence is not zero for all $x_n \in k$.

note: If k is finite $\Rightarrow$ the prop is false

Consider: $p(x) = X^q - X$ on $\mathbb{F}_q = k$.

Example: $I(\{a_1, \ldots, a_n\}) = (X_1 - a_1, \ldots, X_n - a_n)$.

$I(\{a_1, \ldots, a_n\}) = \{ p \in k[X_1, \ldots, X_n] : p(a_1, \ldots, a_n) = 0 \}$

then $X_i - a_i = 0$ at $(a_1, \ldots, a_n)$ $\forall i$

if $x=0$, then $y^2=0 \Rightarrow t=0$

so $(x,y) = (t^2, t^3) \quad \forall (x,y) \in V(I) = V$.

Let $p \in I(V)$

w.t.s. $p = Q(x,y)\,(Y^2 - X^3)$.

$$p \in I(V) = \{ f \in k[x,y] : f(\vec{x}) = 0 \ \forall \vec{x} \in V \}$$
$$= \{ f \in k[x,y] : f(\vec{x}) = 0 \ \forall (x,y) = (t^2, t^3) \}$$

$$= \{ f \in k[x,y] : f(t^2, t^3) = 0 \}$$

$$\Rightarrow p(t^2, t^3) = 0 \quad \forall t \in k.$$

- Divide p by $Y^2 - X^3$ w.r.t. Y.

$$\Rightarrow p = (Y^2 - X^3)\,Q(x,y) + a(X)\,Y + b(X)$$

$$\Rightarrow 0 = 0 + a(t^2)\,t^3 + b(t^2) \quad \forall t \in k$$

i.e. $a(t^2)\,t^3 + b(t^2) = 0 \quad \forall t \in k$

But this can not happen unless the poly.

$$a(T^2)\,T^3 + b(T^2) = 0 \quad \text{in } k[T]$$

$\Rightarrow$ ~~all odd coeff of odd degrees = 0~~

~~all even coeff of even degrees = 0~~

$\Rightarrow a(T^2) = 0 \quad$ i.e. $a(X) = 0$

$b(T^2) = 0 \quad$ i.e. $b(X) = 0$

so $p = (Y^2 - X^3)\,Q(x,y) \in (Y^2 - X^3)$

Hence $I(V) = (Y^2 - X^3)$.

Def: If X is a top. space that satisfies the above conditions, then X is said to be irreducible.

Thm: V = affine alg. set with its Zariski top. Then

V irred $\Leftrightarrow$ $I(V)$ prime $\Leftrightarrow$ $\Gamma(V)$ integral.

note: $\Gamma(V) = \dfrac{k[x_1, \dots, x_n]}{I(V)}$

then $\Gamma(V)$ is integral $\Leftrightarrow$ $I(V)$ is prime.

prf: w.t.s. V irred $\Leftrightarrow$ $I(V)$ prime.

$\Rightarrow$)

Assume V is irred.

Let f, g be two poly. s.t. $fg \in I(V)$

w.t.s. $f \in I(V)$ or $g \in I(V)$.

Since V is affine $\Rightarrow$ $V = V(I(V))$

$= \{ x \in k^n : p(x) = 0 \quad \forall\, p \in I(V) \}$

$\Rightarrow$

$$
* \left\{
\begin{array}{l}
V(f) = \{ x \in k^n : f(x) = 0 \} \\
V(g) = \{ x \in k^n : g(x) = 0 \} \\
V(f) \cup V(g) = \{ x \in k^n : f(x) = 0 \text{ or } g(x) = 0 \}
\end{array}
\right.
$$

But p does not vanish on V.

i.e

$\exists f_i \in I(V_i) - I(V)$ ⊛

then $f_1(x) = 0 \quad \forall\ x \in V_1 \subset V$

$f_2(x) = 0 \quad \forall\ x \in V_2 \subset V$

$\Rightarrow f_1 f_2(x) = 0 \quad \forall\ x \in V =$

b/c $(V = V_1 \cup V_2)$

i.e $f_1 f_2 \in I(V)$

and $I(V)$ is prime. $\Rightarrow f_1 \in I(V)$ or $f_2 \in I(V)$

$\Rightarrow$ contradiction to ⊛.

Corollary:

K is infinite

then K^n is irreducible.

pf:

We know $I(K^n) = (0)$

and (0) is prime (b/c $K[x_1, \dots, x_n]$ is I.D integral domain)

note: If K is finite $\Rightarrow$ coro is false.

K is finite $\Rightarrow K^n$ is finite

$\Rightarrow K^n = \bigcup_{i=1} \{a_i\}$

and $\{a_i\}$ are closed sets.

$\Rightarrow K^n$ is not irred.

Application (Extension of alg. identities):

K is infinite.

V = affine alg. set. $V \neq K^n$.

prf: wts. $\bar{Y}$ is irred.

1- Let $\bar{Y} = F_1 \cup F_2$ where F_i is closed in $\bar{Y}$ $\forall i$

since $\bar{Y}$ is closed in X

$\Rightarrow$ F_i is closed in X $\forall i$

Moreover: $Y = Y \cap \bar{Y}$ $\quad$ $\Downarrow$ $F_i \cap Y$ is closed in Y

$= Y \cap (F_1 \cup F_2)$

$= (Y \cap F_1) \cup (Y \cap F_2)$

since Y is irred

$\Rightarrow$ $Y = Y \cap F_1$ or $Y = Y \cap F_2$

~~say $Y = Y \cap F_1$~~

$\Rightarrow$ $Y \subset F_1$ or $Y \subset F_2$

~~and $F_1 \subset \bar{Y}$~~

$\Rightarrow$ $Y \cup F_2 \subset F_1 \cup F_2$ $\quad$ Assume $Y \subset F_2$

$\Rightarrow$ $F_2 \subset F_1 \cup F_2 = \bar{Y}$

so $F_2 \subset \bar{Y}$ $\quad$ sim. $F_1 \subset \bar{Y}$

$\Rightarrow$ $\bar{Y} = F_2$ or $\bar{Y} = F_1$

then $\bar{Y}$ is irred.

2 = ??

Thm / Def:

V non empty affine alg. set.

we can write V uniquely (up to permutation)

in the form $V = V_1 \cup \cdots \cup V_r$

where V_i is irred affine alg. set. $\forall\ i$.

and $V_i \not\subset V_j$ if $i \neq j$.

~~Deal~~ Terminology: The sets V_i are called the irreducible components of V.

~~tot~~ $p \in k[x_1, \dots, x_n]$. identically.

If $\underbrace{p = 0}$ outside V, then $p \overset{\text{I}}{\equiv} 0$

$\updownarrow$

$p|_{(\bar{a})} = 0 \quad \forall\, \bar{a} \in k^n \setminus V$.

~~[illegible]~~

proposition:

X = top space, $Y \subset X$.

1-* If Y is irred. $\Rightarrow \overline{Y}$ is irred.

* Let $U \subset X$ be open, then

$$\varphi_1 : U \longmapsto \textcircled{U}\, X$$
$$Y \longmapsto \overline{Y} \quad \text{where } Y \text{ is }\overset{\text{irred}}{\text{closed}} \text{ in } U.$$

$$\varphi_2 : X \longrightarrow \textcircled{X}\, U.$$
$$Z \longmapsto Z \cap U \quad \text{where } Z \text{ is }\overset{\text{irred.}}{\text{closed}} \text{ in } X$$

φ_1 & φ_2 are mutually inverse bijections.

$$V(\langle f\rangle \langle g\rangle) = V(f) \cup V(g)$$

But $V = I(V)$

$$\Rightarrow \quad V(I(V)) = V.$$

Moreover, since $fg \in I(V) \Rightarrow \langle fg\rangle \subset I(V)$

$$\Rightarrow \cancel{V \neq} \; \underbrace{V(I(V))}_{= V} \subset V(\langle fg\rangle) = V(f) \cup V(g)$$

$$\Rightarrow \quad V = \big(V(f) \cup V(g)\big) \cap V$$

$$= \big(\underbrace{V(f) \cap V}_{\text{closed in } V}\big) \cup \big(\underbrace{V(g) \cap V}_{\text{closed in } V}\big).$$

V is irred $\Rightarrow$ $V = V(f) \cap V$ or $V = V(g) \cap V$

Assume $V = V(f) \cap V$

$$\Rightarrow V \subset V(f) = \{x \in k^n : f(x) = 0\}$$

$$\Rightarrow f \in I(V) \quad \text{since } f(x) = 0 \quad \forall\, x \in V.$$

ie $f \in I(V)$ or $g \in I(V)$

Hence $I(V)$ is prime.

* Assume $I(V)$ is prime w.t.s. V is irred.

Let $V = V_1 \cup V_2$ where V_1, V_2 are closed in V

Assume $V_i \neq V \quad \forall\, i$.

$V_i \subsetneq V \Rightarrow I(V_i) \supset I(V)$ (I is decreasing)

& $I(V_i) \neq I(V)$ (by Remark 3)

$\Rightarrow \exists$ a poly p s.t. p vanishes on V_i

§ 3 Irreducibility:

~~Consider~~ :

proposition: Let X be a non-empty top. space.

TFAE:

1- if $X = F \cup G$, where F, G are closed in X
then $X = F$ or $X = G$

2- If U, V open in X. & $U \cap V = \varphi$
then $U = \varphi$ or $V = \varphi$.

3- Any non-empty open set of X is dense.

prf: $1 \Leftrightarrow 2$ (easy)

$3 \Rightarrow 2$.

let U, V be two open sets in X. s.t. $U \cap V = \varphi$
w.t.s. $U = \varphi$, or $V = \varphi$

let $U \neq \varphi$, then by (assumption (3))

$\overline{U} = X$.

i.e $\forall x \in X$, any nhd $U_x \cap U \neq \varphi$
$\forall U_x$ nhd of x.

In particular, if $x \in V$
then $V \cap U \neq \varphi \Rightarrow$ contradiction

$2 \Rightarrow 3$

then $(X_1-a_1, \ldots, X_n-a_n) \subset I(\{(a_1,\ldots,a_n)\})$

let $p \in I(\{(a_1,\ldots,a_n)\})$

$\Rightarrow p(a_1,\ldots,a_n)=0$

then By Euclidean division: $Q_i \in K[X_1,\ldots X_n]$

$$p = (X_1-a_1)Q_1 + \cdots + (X_n-a_n)Q_n + c$$

$, c \in K$

And $p(a_1,\ldots a_n)=0 = 0+c \Rightarrow c=0$

Hence

$$p = (X_1-a_1)Q_1 + \cdots (X_n-a_n)Q_n \quad , \in (X_1-a_1,\ldots, X_n-a_n).$$

$\therefore \quad I(\{(a_1,\ldots,a_n)\}) = (X_1-a_1,\ldots, X_n-a_n)$

Example:

let K = infinite.

Calculate: $I(V) = I(V(Y^2-X^3))$

let $I = (Y^2-X^3)$, $V = V(I)$

(by Remark 2) then $I \subset I(V(I)) = I(V(Y^2-X^3)) = I(V)$

So $(Y^2-X^3) \subset I(V)$

Claim: $I(V) \subset (Y^2-X^3)$.

$$V(\overset{(Y^2-X^3)}{\overset{\parallel}{I}}) = \{\vec{x} \in K^2 : Y^2-X^3|_{\vec{x}} = 0\}$$

let $\vec{x} = (x,y) \in V$

$\Rightarrow y^2-x^3=0 \Rightarrow \frac{y^2}{x^2} - x = 0$

Consider $t = y/x$ if $x \neq 0$ $\Rightarrow x = t^2$

$\Rightarrow$ $\Rightarrow t = y/t^2 \Rightarrow y = t^3$

4- we have $I \subset I(V(I))$

I = ideal gen. by $\{p_1, \dots, p_r\}$ i.e $I = \langle p_1, \dots, p_r \rangle$

$V(I) = V(p_1) \cap \dots \cap V(p_r)$.

$I(V(I)) = \{ p \in K[x_1, \dots, x_n] \; ; \; p(x) = 0 \;\; \forall x \in V(I) \}$

But $p_i(x) = 0 \quad \forall x \in V(I)$

then $I \subset I(V(I))$.

* note: Equality does not hold in general:
  - K is not alg. closed $\Rightarrow$ $V(I)$ can be ~~large~~ small!

Ex: $K = \mathbb{R}$

$I = (x^2 + y^2 + 1) \Rightarrow V(I) = \phi$.

$\Rightarrow I(V(I)) = K[x_1, \dots, x_n] \neq I$.

* The map I forgets ~~the~~ powers:

$I = (x^2)$, $n = 2$, $K = \mathbb{R}$

then

$V(I) = \{ x \in \mathbb{R}^2 \; ; \; p(x) = 0 \;\; \forall p \in I \}$

$= \{ x \in \mathbb{R}^2 \; ; \; x^2 = 0 \;\; \forall p \in I \}$ (x,y)

$= \{ (0, y) \}$

$I(V(I)) = \{ f \in \mathbb{R}[x, y] \; ; \; p(0, y) = 0 \}$ ~~example~~

$= (x) \neq I$.

Example: $I(\phi) = K[x_1, \dots, x_n]$.

• Consider $r: K[X_1, \dots, X_n] \longrightarrow \mathcal{F}(V, K)$

$\mathcal{F}(V,K) = \{ f: V \to K \}$. as ring operation addition $f+g$.

then $\ker r = I(V)$

$\text{image } r = \Gamma(V) \cong \dfrac{K[X_1, \dots, X_n]}{I(V)} = \text{ring}$

$\underline{\Gamma(V) \text{ is called the affine algebra of } V}$

Goal: Associate to the geometric object V an alg. object $I(V)$ or $\Gamma(V)$ i.e Translate from ~~the~~ geometric prop $\updownarrow$ alg. properties.

Remarks:

1 – The map I is decreasing.

i.e If $V \subset V_1 \Rightarrow I(V_1) \subset I(V)$.

prf:

let $f \in I(V_1) \Rightarrow f(x) = 0 \quad \forall x \in V_1$

In particular, $\forall x \in V$.

thence $f \in I(V)$

i.e. $I(V_1) \subset I(V)$.

2 – If V is an affine alg. set.

then $\underbrace{V}_{}(I(V)) = V$

the affine alge set defined by $I(V)$

prf: let $x \in V$ w.t.s $x \in V(I(V))$

i.e $p(x) = 0 \quad \forall p \in I(V)$

By def $p \in I(V)$ iff $p(x) = 0 \quad \forall x \in V$

So $x \in V(I(V))$ i.e $V \subset V(I(V))$

Since V is an affine alg. $\Rightarrow V = V(I)$ f.s ideal I

Since $x \notin V(I) \Rightarrow \exists$ at least one f_{i_0} s.t. $f_{i_0}(x) \neq 0$.

But $f_{i_0} g_i \in IJ \quad \forall g_i \in J$

then $f_{i_0} g_i(x) = 0 \quad \forall g_i \in J$

$\Rightarrow g_i(x) = 0 \quad \forall g_i \in J$

So $x \in V(J)$

hence $x \in V(I) \cup V(J)$

i.e. $V(IJ) = V(I) \cup V(J)$.

8\- (5) & (7) $\Rightarrow$ any finite set is an affine alg. set.

Zariski Top:

Ex (1), (6) & (7) $\Rightarrow$ we can define a top on k^n where its closed sets are the affine alg. sets. (we call it the Zariski top.)

Any $X \subset k^n$, inherits an induced top. (Zariski top) with closed set $X \cap \underbrace{V(I)}_{\text{closed in } k^n}$.

Moral: If X is affine alg. set $\Rightarrow$ the closed sets are the aff. alg. sets contained in X.

Note: The sets of the form $V(P_1)$ are called ~~the~~ hyper surfaces.

~~Remark~~ Note: Remark (3) shows that every affine alg. sets ~~com~~ is a finite intersection of hypersurfaces.

4_ Two poly. can define the same affine alg. set.

Example:

$$V(\overset{P_1}{X}) = \{ x \in k^2 : \overset{P(x)}{\cancel{x}} = 0 \}$$
$$= (0, y).$$

$$\|$$

$$V(\overset{P_2 = X^2}{X^2}) = \{ x \in k^2 : P_2(x) = 0 \}$$
$$= (0, y)$$

5- A pt of k^n is an affine alg. set.

Let $(a_1, \dots, a_n) \in k^n$

Let $S = \{ X_1 - a_1, X_2 - a_2, \dots, X_n - a_n \}$

then $V(S) = \{ x \in k^n : X_i - a_i = 0 \quad \forall i \}$

$= (a_1, \dots, a_n)$. b/c $X_1 - a_1 \big|_{a_1} = 0$

$X_2 - a_2 \big|_{a_2} = 0$

$\vdots$

6_ An arbitrary intersection of affine alg. sets is an affine alg. set.

and $\bigcap_j V(S_j) = V(\bigcup_j S_j)$.

$$V(S_j) = \{ x \in k^n : P(x) = 0 \quad \forall \; P \in S_j \}$$

2 - If $n=1$, and $S \neq \{0\}$ i.e $V(S) \neq k$

then $V(S)$ is a finite set (think of $\mathbb{R}$, $\mathbb{C}$)

by def: $V(S) = \{x \in k ;\ p(x) = 0 \ \forall\ p \in S\}$.

This means that the affine alg. sets of a line is the line itself and the finite sets.

Remarks:

1- V is decreasing as a function

i.e $S \subset S_1 \Rightarrow V(S_1) \subset V(S)$.

$V(S_1) = \{x \in k^n :\ p(x) = 0 \ \forall\ p \in S_1\}$

$V(S) = \{x \in k^n :\ p(x) = 0 \ \forall\ p \in S\}$

i.e if $x \in V(S_1) \Rightarrow p(x) = 0 \ \forall\ p \in S_1$

In particular the poly in S.

$\Rightarrow x \in V(S)$

i.e $V(S_1) \subset V(S)$

2- Let $S \subset k[X_1, \dots, X_n]$

~~Denote by~~ $\langle S \rangle$ = ideal generated by S.

i.e $f \in \langle S \rangle$ then $f = \sum_{i=1}^{r} a_i f_i$, $f_i \in S$, $a_i \in k[x_1, \dots, x_n]$

then $V(\langle S \rangle) = V(S)$.

prf: $S \subset \langle S \rangle \Rightarrow V(\langle S \rangle) \subset V(S)$ by 1

w.t.s. $V(S) \subset V(\langle S \rangle)$.