II. Euclidean Space

$\mathbb{R}^n = \{ (x_1, x_2, \ldots, x_n) | x_i \in \mathbb{R} \} \quad \mathbb{R}^1 = \mathbb{R}$

$\mathbb{R}^1 \to \mathbb{R}^2 \not\to \mathbb{R}^3$

For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, define

$||x|| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \left( \sum_{i=1}^{n} x_i^2 \right)^{1/2}$

- This is the norm of $x$, the distance from $x$ to $(0,0,\ldots,0)$

- We define the distance from $x$ to $y$ by the norm of $||x-y||$

  - In $\mathbb{R}^1$, $||x|| = (x^2)^{1/2} = |x|$ (the norm generalizes this)

  - $||x-y|| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2} = \left( \sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2}$

- $\mathbb{R}^n$ with this choice of the distance function is called $n$-dimensional Euclidean space.
Let \( f : D \to \mathbb{R}^n \) where \( D \subseteq \mathbb{R}^m \)
\[ f(x_0) \]
\[ \text{is continuous at } x_0 \text{ when } \forall \varepsilon > 0, \exists \delta > 0, x \in D \text{ and } ||x - x_0|| < \delta \Rightarrow ||f(x) - f(x_0)|| < \varepsilon \]
(Of course \( f \) is cont. if it is cont. at every \( x \in D \))

Text \( f : \mathbb{R} \to \mathbb{R}^2 \) defined by \( f(t) = (\cos(t), \sin(t)) \)

\( f \) is cont.: let \( t_0 \in \mathbb{R} \), let \( \varepsilon > 0 \).
There is a \( \delta_1 > 0 \) so that if \( ||t - t_0|| < \delta_1 \), then \( |\cos(t) - \cos(t_0)| < \frac{\varepsilon}{2} \)
(Since \( \cos \) is continuous)

There is a \( \delta_2 > 0 \) so that if \( ||t - t_0|| < \delta_2 \), then \( |\sin(t) - \sin(t_0)| < \frac{\varepsilon}{2} \)
(Since \( \sin \) is continuous)

So, if \( ||t - t_0|| < \min\{\delta_1, \delta_2\} \), then \( ||f(t) - f(t_0)|| = \sqrt{((\cos(t) - \cos(t_0))^2 + (\sin(t) - \sin(t_0))^2)} \)
\[ = \sqrt{((\cos(t) - \cos(t_0))^2 + (\sin(t) - \sin(t_0))^2)} \]
\[ = \sqrt{\left(\frac{\varepsilon}{2}\right)^2 + \left(\frac{\varepsilon}{2}\right)^2} \]
\[ = \frac{\varepsilon}{\sqrt{2}} < \varepsilon. \]

\( g : \mathbb{R} \to \mathbb{R}^2 \) \( g(t) = (g_1(t), g_2(t)) \)
\( x = g_1(t) \)
\( y = g_2(t) \)

Notice that this argument would work for any two continuous functions in the role of \( \cos \) and \( \sin \).

For \( 1 \leq k \leq n \), \( P_k : \mathbb{R}^n \to \mathbb{R} \) denotes the projection function defined by \( P_k(x) = k \)
\[ P_k((x_1, x_2, \ldots, x_n)) = x_k \]
Let \( f: \mathbb{R}^m \to \mathbb{R}^n \) be a function.

For \( x \in \mathbb{R}^m \), \( f(x) = (f_1(x), f_2(x), \ldots, f_n(x)) \).

Write \( f_k : \mathbb{R}^m \to \mathbb{R} \) for the function \( f_k \circ \).

This is called the \( k \)-th coordinate function of \( f \).

Then, \( f(x) = (f_1(x), f_2(x), \ldots, f_n(x)) \).

An \( f \) from \( f : \mathbb{R}^m \to \mathbb{R}^n \) determines \( n \) coordinate functions \( f_k : \mathbb{R}^m \to \mathbb{R} \).

On the other hand, if we start with \( n \) functions, \( g_1, \ldots, g_n : \mathbb{R}^m \to \mathbb{R} \), we can define a function \( g : \mathbb{R}^m \to \mathbb{R}^n \) by \( g(x) = (g_1(x), g_2(x), \ldots, g_n(x)) \).

Example: \( f : \mathbb{R}^2 \to \mathbb{R}^2 \)

\[ f(x, y) = x + iy \]

\[ f(z) = z^2 \]

Coordinate functions of \( f \):

\[ f_1(x, y) = (x + iy)^2 = x^2 + 2xy + iy \]

\[ f_2(x, y) = x^2 - y^2 \] and \( f_2(x, y) = 2xy \)

---

**Homework**

So if \( f \)

\[ |f(x)-f(x)| = \varepsilon |g(x_0)| \]

in which case

\[ |g(x_0)| = |g(x_0) - g(x)| + |g(x) - g(x)| \]

\[ = \frac{1}{2} |g(x_0)| + |g(x)| \]

\[ |g(x)| = \frac{1}{2} |g(x_0)| \]
\[ \pi_k : \mathbb{R}^n \to \mathbb{R} \quad \pi_k((x_1, \ldots, x_n)) = x_k \]

\[ f : \mathbb{R}^m \to \mathbb{R}^n \quad f((x_1, \ldots, x_m)) = (\pi_1 \circ f)(x_1, \ldots, x_m), (\pi_2 \circ f)(x_1, \ldots, x_m) \]

\[ f_i : \mathbb{R}^m \to \mathbb{R} \]

- Each \( \pi_k \) is continuous (HW)
  - So given \( \varepsilon \), choosing \( \delta = \varepsilon \) should work.

- Consequently, if \( f : \mathbb{R}^m \to \mathbb{R}^n \) is continuous, then each \( \pi_k \circ f \) is a composition of continuous functions (HW)
- Conversely, if each \( \pi_k \) is continuous, then \( f \) is continuous.

**Example**: \( \mathbb{R} \to \mathbb{R}^2, t \mapsto (\cos t, \sin t) \) is continuous because cosine and sine are continuous.

**Open Balls & Open Sets**

**Definition**: Let \( x \) be a point in \( \mathbb{R}^n \), let \( \varepsilon > 0 \)

\[ B(x, \varepsilon) = \{ z \in \mathbb{R}^n \mid \| x - z \| < \varepsilon \} \]

This is the open ball with radius \( \varepsilon \) and center \( x \)

**Proposition**: Let \( W \subseteq \mathbb{R}^n \). Then \( \forall x \in W, \exists \varepsilon > 0, B(x, \varepsilon) \subseteq W \iff W \) is a union of open balls.

**Note**: Not possible for a closed square

**Note**: Also not possible if \( W \) is a subset of \( \mathbb{R}^n \).
\[ \text{Proof:} \] Suppose \( x \in W \), \( \exists \varepsilon > 0 \), \( B(x, \varepsilon) \subseteq W \)

For each \( x \in W \), choose a specific value \( \varepsilon > 0 \) so that the ball around \( x \) with radius \( \varepsilon \) is contained in \( W \).

We will show that \( W = \bigcup_{x \in W} B(x, \varepsilon) \).

Suppose \( \varepsilon \in W \) and \( \exists \ v \in B(\varepsilon, \varepsilon) \),

\[ \therefore \varepsilon \in \bigcup_{x \in W} B(x, \varepsilon) \]

Suppose \( \varepsilon \in \bigcup_{x \in W} B(x, \varepsilon) \),

Then \( \varepsilon \in B(y, \varepsilon) \) for some \( y \in W \).

\[ B(y, \varepsilon) = W \therefore \varepsilon \in W \quad \text{(since each } x \in W \text{ is open in } W) \]

Better way: \[ W = \bigcup_{x \in W} B(x, \varepsilon) \subseteq \bigcup_{x \in W} B(x, v) \subseteq W \text{(since each } B(x, v) \subseteq W) \]

("\(\Rightarrow\)"") Conversely, suppose \( W \) is a union of open balls, say \( W = \bigcup_{x \in A} B(x, \varepsilon) \).

Let \( x \in W \). Then \( x \in B(x, \varepsilon) \) for some \( x \in A \).

Put \( \delta = \varepsilon - \|x - x_0\| \)

\( \varepsilon > 0 \) since \( x \in B(x, \varepsilon) \) implies \( \|x - x_0\| < \varepsilon \).

Let \( \varepsilon \in W \), then \( \varepsilon \in B(x, \varepsilon) \).

\( \|z - x\| = \|z - x + x - x_0\| \leq \|z - x_0\| + \|x - x_0\| \leq (\varepsilon - 1) + \|x - x_0\| = \varepsilon \)

\[ \therefore z \in B(x, \varepsilon) \therefore \varepsilon \in W \]

\[ \therefore B(x, \varepsilon) \subseteq W. \]
Define a subset $U \subseteq \mathbb{R}^n$ is called open when $\forall x \in U, \exists \epsilon > 0, B(x, \epsilon) \subset U$ or equivalently, it is a union of open balls $B(x, \epsilon)$. Notice also that $\#(B(x_0, s)) = B(x_0, 2s)$ is the same as saying $\#(B(x_0, s)) = \text{open}$ and $B(x_0, s)$ is also the open ball where $\#(B(x_0, s)) = B(x_0, s)$.

So, $\exists \lim_{x \to x_0, s > 0} \psi_B(x_0, s) = 0$. In the open ball where $\#(B(x, s)) 

\|a - b \| = \|a - c + c - b\| = \|a - c\| + \|c - b\|$. Note: $\#(B(x_0, s)) = B(x_0, 2s)$ means $\text{B(x_0, 2s)}$.
Theorem. \( f: \mathbb{R}^m \rightarrow \mathbb{R}^n \) the following 3 things are equivalent:

1. \( \forall x, \in \mathbb{R}^m \) and \( \forall \varepsilon > 0, \exists S > 0, \text{ s.t. } \|x-x_0\| < S \Rightarrow \|f(x) - f(x_0)\| < \varepsilon \)

2. For every open set \( U \subseteq \mathbb{R}^n \), \( f^{-1}(U) \) is open in \( \mathbb{R}^m \)

3. For every open ball \( B(y, \varepsilon) \) in \( \mathbb{R}^n \), \( f^{-1}(B(y, \varepsilon)) \) is open in \( \mathbb{R}^m \)

Proof:

1. \( \implies \) 2

Assume 1 is true.

Let \( U \) be an open set in \( \mathbb{R}^n \), let \( x \) be an element of \( f^{-1}(U) \), so \( f(x) \in U \)

By 1, \( \exists S > 0 \text{ s.t. } f(B(x, S)) \subseteq B(f(x), \varepsilon) \subseteq U \)

i.e. \( f(B(x, S)) = f^{-1}(B(f(x), \varepsilon)) \subseteq f^{-1}(U) \)

\( \therefore f^{-1}(U) \) is open

2. \( \implies \) 3

Let \( B(y, \varepsilon) \) be an open ball.

\( B(y, \varepsilon) \) is open (since it's a union of 1 open ball), so by 2, \( f^{-1}(U) \) is open.

3. \( \implies \) 1

Assume, 3, let \( x_0 \in \mathbb{R}^m \) and let \( \varepsilon \) be given.

\( f(x_0) \in B(f(x_0), \varepsilon) \), so \( x_0 \in f^{-1}(B(f(x_0), \varepsilon)) \)

By 3 this is open \( \exists S > 0 \text{ s.t. } B(x_0, S) \subseteq f^{-1}(B(f(x_0), \varepsilon)) \)

\( \therefore f(B(x_0, S)) \subseteq B(f(x_0), \varepsilon) \)

If \( \|x-x_0\| < S \), then \( \|f(x) - f(x_0)\| < \varepsilon \). \( \square \)
$w$ is open if $\forall x \in w, \exists \varepsilon > 0,$

$q \subseteq \mathbb{R}$ not open $p_\alpha \not\subseteq p_\alpha + \varepsilon$
contains an irrational, so $p_\alpha \not\subseteq q$
contains a rational $p_\alpha + \varepsilon$

$-R - q$ is also not open for the same reason.

- No open balls fit inside the graph, these points not in $w$.

but the complement, $\mathbb{R}^2 - \mathcal{S}$ is open.

$f: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is continuous (in the $\varepsilon - \delta$ def.)

$\iff \forall U \text{open in } \mathbb{R}^n, f^{-1}(U) \text{ is open in } \mathbb{R}^m$

**Properties of Open Sets in $\mathbb{R}^n$**

1. Let $\mathcal{U}_\mathcal{A}$ be a collection of open sets in $\mathbb{R}^n$. Then $\bigcup \mathcal{U}_\mathcal{A}$ is open.

   **Proof:** let $x \in \bigcup \mathcal{U}_\mathcal{A}.$ Then $x \in U_\alpha$ for some $\alpha \in \mathcal{A}$. $U_\alpha$ is open, so $\exists \varepsilon > 0, B(x, \varepsilon) \subseteq U_\alpha \subseteq \bigcup \mathcal{U}_\mathcal{A}$.

2. Let $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_k$ be a finite collection of open subsets of $\mathbb{R}^n$. Then $\bigcap \mathcal{U}_i$ is open.

   **Proof:** let $x \in \bigcap \mathcal{U}_i$. For each $i, x \in \mathcal{U}_i$ and $\mathcal{U}_i$ is open, so $\exists \varepsilon_i$ s.t. $B(x, \varepsilon_i)$ is contained in $\mathcal{U}_i$.

   Let $\varepsilon = \min \varepsilon_1, \ldots, \varepsilon_k.$ For each $i, \varepsilon \leq \varepsilon_i$ so $B(x, \varepsilon) \subseteq B(x, \varepsilon_i) \subseteq \bigcap \mathcal{U}_i$.

   **Note:** (2) can fail for infinitely many open sets.

   $\bigcap_{i=1}^\infty (-\frac{1}{i} , \frac{1}{i}) = \emptyset$

   open not open

   Similarly, $\bigcap_{i=1}^\infty B(x, \frac{1}{i}) = \emptyset \times \mathcal{S}$
For all \( x \in \mathbb{R}^n \), \( \mathbb{R}^n - \{x\} \) is open.

**Proof:** Let \( x \in \mathbb{R}^n - \{x\} \), i.e., \( x \neq x \).

\[ ||x - \frac{1}{2}x|| > 0, \quad \text{so} \quad B\left(\frac{1}{2}x, \frac{1}{2}||x - x||\right) \subset \mathbb{R}^n - \{x\} \]

(for if not, then it would have to contain \( x \), so \( ||x - \frac{1}{2}x|| < \frac{1}{2}||x - x|| \).

\[ ||x - \frac{1}{2}x|| < c \] for a contradiction.)

Let \( \mathbb{R}^n \) be a finite collection of points in \( \mathbb{R}^n \). Then \( \mathbb{R}^n - \mathbb{K} \) is open.

**Proof:**

\[ \mathbb{R}^n - \mathbb{K} = \mathbb{R}^n - \bigcup_{i=1}^{K} \mathbb{R}^n - \{x_i\} = \bigcap_{i=1}^{K} \mathbb{R}^n - \{x_i\} \quad \text{(De Morgan's law: if } \bigcup_{i=1}^{K} U_i \text{ and } U_j \quad \text{then } U_j - \bigcup_{i=1}^{K} U_i = \bigcap_{i=1}^{K} (U_j - U_i)) \]

This is an intersection of finitely many open sets, so it is open.

**Note:** Every subset of \( \mathbb{R}^n \) is an intersection of some collection of open sets.

**Proof:** Let \( A \subseteq \mathbb{R}^n \). \( \mathbb{R}^n - \mathbb{A} \) is open (the collection of open sets is \( \{ \mathbb{R}^n - x \} \) if \( x \in \mathbb{R}^n - \mathbb{A} \).)

by De Morgan's law,

\[ \mathbb{R}^n - \bigcup_{x \in \mathbb{R}^n - \mathbb{A}} x = \mathbb{R}^n - \mathbb{A} \]

**Claim:** \( \mathbb{R}^2 - \Gamma \) is open.

**Proof:** Let \( (x, y) \in \mathbb{R}^2 - \Gamma \).

Since \( \sin \) is continuous, there exists \( \delta > 0 \) s.t. if \( ||x - x_0|| < \delta \), then \( ||\sin(x) - \sin(x_0)|| < \frac{||y_0 - \sin(x_0)||}{2} \).

Radius must be smaller than \( \frac{||y_0 - \sin(x_0)||}{2} \).

Put \( \varepsilon = \min\{ \delta, \frac{||y_0 - \sin(x_0)||}{2} \} \).

Now need to show the ball is in \( \mathbb{R}^2 - \Gamma \) — contradiction.

Suppose \( B(x_0, \varepsilon) \cap \Gamma \neq \emptyset \).

Then \( \exists x \) with \( (x, \sin(x)) \in B(x_0, y_0, \varepsilon) \).

\[ ||x - x_0|| \leq \varepsilon \leq \delta \quad \text{so supposing in ball} \]

\[ ||\sin(x) - \sin(x_0)|| \leq \frac{||y_0 - \sin(x_0)||}{2} \]
\[ \left| y_0 - \sin(x) \right| \leq \left| (x, \sin(x)) - (x_0, y_0) \right| < \varepsilon = \frac{\left| y_0 - \sin(x_0) \right|}{2} \]
\[
\left| y_0 - \sin(x_0) \right| \leq \left| y_0 - \sin(x) \right| + \left| \sin(x) - \sin(x_0) \right| < \frac{\left| y_0 - \sin(x_0) \right|}{2} + \frac{\left| y_0 - \sin(x_0) \right|}{2} = \left| y_0 - \sin(x_0) \right| = \frac{\left| y_0 - \sin(x_0) \right|}{2}.
\]

Contradiction: The number is less than itself.

\[ \therefore B((x_0, y_0), \varepsilon) \subseteq \mathbb{R}^2 - \Gamma \]

Define \( F : \mathbb{R}^2 \to \mathbb{R} \) by \( F(x,y) = y - \sin(x) \leq \text{continuous} \)

\[ F(x,y) = 0 \iff y = \sin(x) \iff (x,y) \in \Gamma \]

so \( \Gamma = F^{-1}(\{0\}) \), so \( \mathbb{R}^2 - \Gamma = F^{-1}(\mathbb{R} - \{0\}) \) open.

Note: Can generalize both of these to show the graph of any continuous function \( f : \mathbb{R} \to \mathbb{R} \) is \( \text{0-cont.} \)

\( \text{If } g : X \to Y \text{ is a function, } X \times Y \text{ is the set of ordered pairs where } \{ (x,y) \mid x \in X, y \in Y \} \)

The graph of \( g \) is \( \Gamma_g = \{ (x, g(x)) \mid x \in X \} \)

Prop: \( \text{If } g \text{ is a continuous function between topological spaces, then } X \times Y - \Gamma_g \text{ is always an open set} \)

\( \text{Let: generalizes to prove this, but } \square \text{ doesn't because } x - g(x) \text{ doesn't make sense unless } Y \text{ has a subtraction operation} \)

\( \text{?; if } f : \mathbb{R} \to \mathbb{R} \text{ and } \mathbb{R}^2 - \Gamma \text{ is open, must } f \text{ be cont.?} \)
motivation for def of Top. Space: (Basic Ideas)
continuity for maps $f: \mathbb{R}^m \to \mathbb{R}^n$
\[ \varepsilon \to \delta \text{ definition} \]

reformed to minimize the role of distance (by thinking about sets instead of distances)
For $x \in \mathbb{R}^k$, define $B(x, \varepsilon) = \{ z \in \mathbb{R}^k : \| z - x \| \leq \varepsilon \}$

define $U \subseteq \mathbb{R}^k$ to be open if...
(equivalently, $U$ is open when $U$ is a union of open balls)
$f$ satisfies the $\varepsilon-\delta$ def. $\iff \forall U$ open in $\mathbb{R}^n$,
$f^{-1}(U)$ is open in $\mathbb{R}^m$

Properties of Open Sets in $\mathbb{R}^n$: $\iff$ General def. of the open sets of a space
$f$ is cont. when $\forall U \text{ open}$, $f^{-1}(U)$ is open

General def. of cont. of $f: X \to Y$

Open set in $\mathbb{R}^n$ is a "basis" of sets that generate the topology
- every open set is a union of basic open sets

Standard Topology on $\mathbb{R}^n$ derives from the topology on $\mathbb{R}$
- a basis is $\{ (a, b) \times (a_2, b_2) \times \cdots \times (a_n, b_n) \}$

rectangles on $\mathbb{R}^n$

Properties of distance in $\mathbb{R}^n$: "product topology" on $X \times Y$ when $X$ and $Y$ have topologies

General concept of a metric $d$ (HW 10)

gives a metric topology—but not all topologies come from metrics