Mathematics 5863 Name (please print) ____________________

Examination II

April 7, 2005

Instructions: Give brief, clear answers, avoiding excessive details (especially when the instruction is "Verify"). For example, if $F\colon \alpha \simeq_p \beta$ is a path homotopy, you need not verify that $f \circ F$ is a path homotopy from $f \circ \alpha$ to $f \circ \beta$. If you lift a path homotopy, you need not verify that the lifted homotopy is a path homotopy.

Even in problems where you cannot do one of the early parts of the problems, try to solve later parts of the problem taking the previous parts as known.

I. (15) Let X be a path-connected space. In this question, you may assume the facts that if each $\alpha_i \simeq_p \alpha_i'$, then $\alpha_1 * \cdots * \alpha_n \simeq_p \alpha_1' * \cdots * \alpha_n'$, and that $c_{\alpha(0)} * \alpha \simeq_p \alpha \simeq_p \alpha * c_{\alpha(1)}$.

(a) Suppose that γ is a path in X. Define the change-of-basepoint function $h_\gamma\colon \pi_1(X, \gamma(1)) \to \pi_1(X, \gamma(0))$.

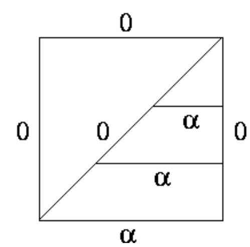
(b) Verify that if τ is another path in X, with $\tau(0) = \gamma(1)$, then $h_{\gamma * \tau} = h_\gamma \circ h_\tau$.

(c) Verify that if $\gamma \simeq_p \gamma'$, $h_\gamma = h_{\gamma'}$.

(d) Verify that $h_{c_{x_0}} = id_{\pi_1(X, x_0)}$, where as usual c_{x_0} denotes the constant path at a point $x_0 \in X$.

(e) From the previous facts, deduce that h_γ is bijective.

II. (5) Let $\alpha\colon I \to \mathbb{R}$ be the loop at 0 defined by $\alpha(t) = \sin(\pi t)$. Show that the function $F\colon I \times I \to \mathbb{R}$ indicated by the diagram shown at the right is not continuous.



III. (5) Let $f\colon (X, x_0) \to (Y, y_0)$ be a map taking basepoint to basepoint. Show that the induced homomorphism $f_\#\colon \pi_1(X, x_0) \to \pi_1(Y, y_0)$ defined by $f_\#(\langle \alpha \rangle) = \langle f \circ \alpha \rangle$ is well-defined.

IV. (10) Let $F = T\#D$ be the orientable surface of genus 1 with one boundary circle, let $x_0 \in \partial F$, and let $i\colon \partial F \to F$ denote the inclusion. We know that $\pi_1(\partial F, x_0) \cong \mathbb{Z}$, since $\partial F \approx S^1$.

(a) Give an explanation (the right picture would be very helpful) of why $\pi_1(F, x_0)$ contains two elements a and b for which $aba^{-1}b^{-1} = i_\#(c)$ where c generates $\pi_1(\partial F, x_0)$.

(b) Use the fact in (a) to prove that ∂F is not a retract of F.

V. (15) Recall that a simply-connected space is path-connected and has the property that any two paths with the same starting and ending points are path homotopic. Let X be a simply-connected space with basepoint x_0, let $f\colon X \to S^1$ be any continuous map, and suppose that $r_0 \in \mathbb{R}$ with $p(r_0) = f(x_0)$. Define $\widetilde{f}\colon X \to \mathbb{R}$ as follows: For each $x \in X$, choose a path $\alpha\colon I \to X$ from x_0 to x, let $\widetilde{f \circ \alpha}\colon I \to \mathbb{R}$ be the unique lift of $f \circ \alpha$ starting at r_0, and define $\widetilde{f}(x) = \widetilde{f \circ \alpha}(1)$.

(a) Verify that $\widetilde{f}$ is well-defined.

(b) Verify that $p \circ \widetilde{f} = f$.

(c) Take as known the fact that $\widetilde{f}$ is continuous, that is, that $\widetilde{f}$ is a lift of f. Prove that if $n \geq 2$ then any continuous map $f\colon S^n \to S^1$ is homotopic to a constant map.

VI. (10) Let $p\colon \mathbb{R} \to S^1$ be the usual covering map $p(r) = e^{2\pi i r}$. Let $s_0 = (1,0) = p(0) \in S^1$. Define $\Phi\colon \pi_1(S^1, s_0) \to \mathbb{Z}$ by $\Phi(\langle\alpha\rangle) = \widetilde{\alpha}(1)$, where $\widetilde{\alpha}$ is the lift of α to $\mathbb{R}$ starting at 0 (take as known the fact that Φ is well-defined).

(a) Prove that Φ is injective.

(b) Prove that Φ is surjective.

VII. (15) Let $f\colon S^1 \to S^1$ be a continuous map.

(a) Define the *degree* of f.

(b) Verify that the map $f_n\colon S^1 \to S^1$ defined by $f_n(z) = z^n$ has degree n.

(c) Taking as known the fact that the degree of a composition is the product of degrees, verify that if f is a homeomorphism, the f has degree 1 or -1.

(d) Taking as known the fact that two maps from S^1 to S^1 are homotopic to if and only if they have the same degree, prove that if $f\colon S^1 \to S^1$ is homotopic to a homeomorphism, then f is homotopic to exactly one of the maps id_{S^1} or f_{-1}.