Mathematics 5863 Name (please print) ____________________

Final Examination

May 10, 2005

Give brief answers, emphasizing the key point without using too much time on unimportant details.

Even in problems where you cannot do one of the parts, try to solve later parts of the problem taking the previous parts as known.

In any problem involving fundamental groups or covering maps, it is assumed that all spaces involved are connected, locally path-connected, semilocally simply-connected Hausdorff spaces.

You may freely use the fact that $\chi(S^2\#gT\#nP\#kD) = 2 - 2g - n - k$.

I. (8) State the Invariance of Domain theorem. Prove that if $j\colon S \to \mathbb{R}^2$ is an imbedding from a 2-manifold S with empty boundary to $\mathbb{R}^2$, the the image of j is open.

II. (8) Let $A \subseteq B \subseteq X$.

1. Prove that if A is a retract of X, then A is a retract of B.

2. Find an example where A is a deformation retract of X, but A is not a deformation retract of B.

III. (6) Give an example (with a brief explanation) of two compact, connected surfaces that are homotopy equivalent but not homeomorphic.

IV. (10) All surfaces in this problem are assumed to be compact and connected. As usual, $\chi(F)$ means the Euler characteristic of F.

1. Let F be a compact, connected surface. Recall that when one removes the interior of a 2-disk admissibly imbedded in F, the resulting manifold is $F\#D$. Use the Classification Theorem to verify that $\chi(F \# D) = \chi(F) - 1$.

2. Let F_1 and F_2 be two surfaces, and let F be a connected surface obtained by identifying a boundary circle of F_1 with a boundary circle of F_2 (for example, when one identifies the boundaries of two disks, one obtains a 2-sphere). Use the Classification Theorem to verify that $\chi(F) = \chi(F_1) + \chi(F_2)$.

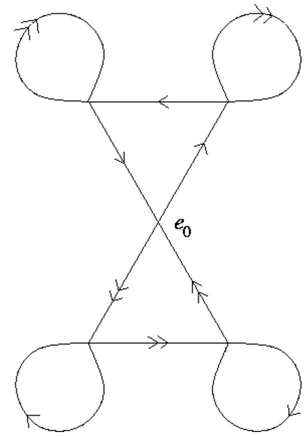
3. Put these two items of information together with the fact that $F_1\#F_2$ is obtained by identifying boundary circles of $F_1\#D$ and $F_2\#D$ to prove that $\chi(F_1 \# F_2) = \chi(F_1) + \chi(F_2) - 2$.

V. (6) Assume the fact that if a map $f\colon X \to Y$ is homotopic to a constant map, then it extends to a map $F\colon C(X) \to Y$, where $C(X)$ is the cone on X, and X is regarded as a subspace of $C(X)$ by identifying x with $(x, 0)$. Prove that S^2 is not contractible.

VI. (6) Let X and Y be spaces, and give the set of continuous functions $\mathcal{C}(X, Y)$ the compact-open topology. Show that if Y is Hausdorff, then $\mathcal{C}(X, Y)$ is Hausdorff.

VII. (6) Give the set of continuous functions $\mathcal{C}(\mathbb{R}, \mathbb{R})$ from $\mathbb{R}$ to $\mathbb{R}$ the compact-open topology. For $n \in \mathbb{N}$, define $f_n \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ by $f_n(x) = 0$ for $x \le n$, $f_n(x) = x - n$ for $n \le x \le n + 1$, and $f_n(x) = 1$ for $n + 1 \le x$. Prove that the sequence $\{f_n\}$ converges to the zero function.

VIII. (8) A covering space E of $B = S^1 \vee S^1$ is shown at the right. As usual, the single arrows cover the circle which corresponds to an element $a \in \pi_1(B, b_0)$, and the double arrows cover the other circle, which corresponds to an element $b \in \pi_1(B, b_0)$. Write G for $\pi_1(B, b_0)$ and H for $p_\#(\pi_1(E, e_0))$.



1. Determine the number n of right cosets of H in G, and find elements $g_1, \ldots, g_n$ in G such that the cosets are $Hg_1, \ldots, Hg_n$.
2. What is the group of covering transformations for this covering space? Why?

IX. (6) Let X, Y, and Z be locally compact Hausdorff spaces, and let $C\colon \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z)$ be the composition function defined by $C(f,g) = g \circ f$. Show that if $g \circ f \in S(C,U) \subseteq \mathcal{C}(X,Z)$, then there exists a basic open set W in $\mathcal{C}(X,Y) \times \mathcal{C}(Y,Z)$ such that $(f,g) \in W$ and $C(W) \subseteq S(C,U)$.

X. (8) Let $X = \cup_{i=1}^{\infty} U_i$, where each U_i is open and each $U_i \subseteq U_{i+1}$.

1. Prove that if $\alpha\colon (I, \partial I) \to (X, x_0)$ represents an element of $\pi_1(X, x_0)$, then $\alpha(I) \subseteq U_N$ for some N.
2. Prove that if each U_i is simply-connected, then X is simply-connected.

XI. (6) State the Lifting Criterion for covering maps.

XII. (10) Let $p\colon (E, e_0) \to (B, b_0)$ be a covering map. Prove that a loop α in B based at b_0 lifts to a loop in E at e_0 if and only if $\langle \alpha \rangle \in p_\#(\pi_1(E, e_0))$.

XIII. (6) Let $p\colon \widetilde{X} \to X$ and $q\colon \widetilde{Y} \to Y$ be covering maps with $\widetilde{X}$ and $\widetilde{Y}$ simply-connected. Prove that if $f\colon X \to Y$ is any map, then there exists a map $F\colon \widetilde{X} \to \widetilde{Y}$ such that $q \circ F = f \circ p$.