Math 5863 homework

35. (3/22) Recall that if H is a subgroup of a group G, then gHg^{-1} is the subgroup consisting of all elements ghg^{-1} for $h \in H$, and recall that H is called a *normal* subgroup if $gHg^{-1} = H$ for all $g \in G$.

    1. Verify that every subgroup of an abelian group is normal.
    2. Verify that the subgroup consisting of the powers of α is a normal subgroup of D_n.
    3. Verify that the subgroup consisting of the powers of β is a normal subgroup of D_n if and only if $n \leq 2$.
    4. Let T be the subgroup of $\mathrm{Isom}_+(\mathbb{R}^2)$ consisting of all translations, that is, all elements of the form T_v. Verify that T is isomorphic to $\mathbb{R}^2$, and is a normal subgroup of $\mathrm{Isom}_+(\mathbb{R}^2)$.
    5. Let R be the subgroup of $\mathrm{Isom}_+(\mathbb{R}^2)$ consisting of all rotations, that is all elements of the form R_α. Verify that R is isomorphic to S^1, and is not a normal subgroup of $\mathrm{Isom}_+(\mathbb{R}^2)$.
    6. Find a subgroup of $\mathrm{Isom}(\mathbb{R}^2)$ (not $\mathrm{Isom}_+(\mathbb{R}^2)$, as you will want to use the isometry $\tau(x, y) = (x, -y)$) that is isomorphic to D_n.

36. (3/22) Consider the quotient space of the standard 2-sphere S^2 in $\mathbb{R}^3$, obtained by identifying each x with $-x$.

    1. Show that the quotient space is homeomorphic to the real projective plane $P = \mathbb{RP}^2$, obtained from a Möbius band and a 2-disk by identifying their boundary circles.
    2. Let $p \colon S^2 \to P$ be this quotient map. Show (a good picture should be enough) that each $x \in P$ has an open neighborhood U for which $p^{-1}(U)$ consists of two copies of U, each mapped homeomorphically to U by the restriction of p.
    3. The previous condition implies that $p \colon S^2 \to P$ satisfies the unique path lifting and unique homotopy lifting theorems, just as with the map $\mathbb{R} \to S^1$ (no need to prove this, the argument is exactly the same). Use these to prove that $\pi_1(P) \cong C_2$ (the fact that S^2 is simply-connected, which we proved in class since we proved that $\pi_1(S^2) = \{1\}$, is needed in the argument).

37. (4/5) Let X and Y be spaces, and give the set of continuous functions $\mathcal{C}(X, Y)$ the C-O topology. Show that if Y is Hausdorff, then $\mathcal{C}(X, Y)$ is Hausdorff.

38. (4/5) Let X and Y be spaces, and give the set of continuous functions $\mathcal{C}(X, Y)$ the C-O topology. Prove that if A is a subspace of X, then the function $\mathcal{C}(X, Y) \to \mathcal{C}(A, Y)$ determined by sending f to $f|_A$ is continuous.