Math 5863 homework solutions

26. (3/8) Let $\alpha\colon I \to S^1$ be a path. Let $\widetilde{\beta}_1$ and $\widetilde{\beta}_2$ be two lifts of α to $\mathbb{R}$. Prove that for some $N \in \mathbb{Z}$, $\widetilde{\beta}_2(t) = \widetilde{\beta}_1(t) + N$ for all $t \in I$ (let $N = \widetilde{\beta}_2(0) - \widetilde{\beta}_1(0)$ and define $\tau(r) = r + N$, check that $p \circ \tau = p$, and use uniqueness of path lifting). Deduce that $\widetilde{\beta}_1(1) - \widetilde{\beta}_1(0) = \widetilde{\beta}_2(1) - \widetilde{\beta}_2(0)$.

    Let $N = \widetilde{\beta}_2(0) - \widetilde{\beta}_1(0)$. Since $p(\widetilde{\beta}_1(0) - \widetilde{\beta}_1(0)) = \exp(2\pi i\widetilde{\beta}_2(0) - 2\pi i\widetilde{\beta}_1(0)) = \exp(2\pi i\widetilde{\beta}_2(0))/\exp(2\pi i\widetilde{\beta}_1(0) = p(\widetilde{\beta}_2(0))/p(\widetilde{\beta}_1(0)) = \alpha(0)/\alpha(0) = 1$, N is an integer. Define $\tau\colon \mathbb{R} \to \mathbb{R}$ by $\tau(r) = r + N$, so that $p\circ\tau(r) = p(r+N) = p(r)$, that is, $p \circ \tau = p$. We have $p \circ \tau \circ \widetilde{\beta}_1 = p \circ \widetilde{\beta}_1 = \alpha$, and $\tau \circ \widetilde{\beta}_1(0) = \widetilde{\beta}_2(0)$, so by uniqueness of lifts, $\tau \circ \widetilde{\beta}_1 = \widetilde{\beta}_2$, that is, $\widetilde{\beta}_2(t) = \widetilde{\beta}_1(t) + N$ for all t. In particular, for $t = 1$ we have $\widetilde{\beta}_2(1) = \widetilde{\beta}_1(1) + \widetilde{\beta}_2(0) - \widetilde{\beta}_1(0)$, so $\widetilde{\beta}_1(1) - \widetilde{\beta}_1(0) = \widetilde{\beta}_2(1) - \widetilde{\beta}_2(0)$.

27. (3/8) Prove that $q\colon \mathbb{Z} \times \mathbb{R} \to S^1$ defined by $q(n, r) = p(r)$ has unique path lifting and unique homotopy lifting. (Let $\alpha\colon I \to S^1$ and let $(n, r_0) \in \mathbb{Z} \times \mathbb{R}$ with $q(n, r_0) = \alpha(0)$. By unique path lifting for $\mathbb{R} \to S^1$, there exists $\widetilde{\alpha}_1\colon I \to \mathbb{R}$ with $p \circ \widetilde{\alpha}_1(t) = \alpha(t)$. Use $\widetilde{\alpha}_1$ to define the lift $\widetilde{\alpha}$. To prove that the $\widetilde{\alpha}$ is unique, let $p_1\colon \mathbb{Z} \times \mathbb{R} \to \mathbb{Z}$ and $p_2\colon \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ be the projection maps, and show that $p_1 \circ \widetilde{\alpha}$ and $p_2 \circ \widetilde{\alpha}$ are uniquely determined.)

    Let $\alpha\colon I \to S^1$ and let $(n, r_0) \in \mathbb{Z}\times\mathbb{R}$ with $q(n, r_0) = \alpha(0)$. By unique path lifting for $\mathbb{R} \to S^1$, there exists $\widetilde{\alpha}_1\colon I \to \mathbb{R}$ with $p \circ \widetilde{\alpha}_1(t) = \alpha(t)$. Define $\widetilde{\alpha}\colon I \to \mathbb{Z} \times \mathbb{R}$ by $\widetilde{\alpha}(t) = (n, \widetilde{\alpha}_1(t))$; this is a lift of α. To prove that it is unique, suppose that $\widetilde{\gamma}(t)\colon I \to \mathbb{Z} \times \mathbb{R}$ is any lift of α starting at (n, r_0). Let $p_1\colon \mathbb{Z} \times \mathbb{R} \to \mathbb{Z}$ and $p_2\colon \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ be the projection maps. Now $p_1 \circ \widetilde{\gamma}\colon I \to \mathbb{Z}$, and since the maximal connected subsets of $\mathbb{Z}$ are points, $p_1 \circ \widetilde{\gamma}$ must be the constant map to n. On the other hand, $p_2 \circ \widetilde{\gamma}\colon I \to \mathbb{R}$, and $p \circ p_2 \circ \widetilde{\gamma} = q \circ \widetilde{\gamma} = \alpha$, so by uniqueness of lifts to $\mathbb{R}$, $p_2 \circ \widetilde{\gamma} = \widetilde{\alpha}_1$. Since $p_1 \circ \widetilde{\gamma} = p_1 \circ \widetilde{\alpha}$ and $p_2 \circ \widetilde{\gamma} = p_2 \circ \widetilde{\alpha}$, we have $\widetilde{\gamma} = \widetilde{\alpha}$. The proof for unique lifting of homotopies is very similar.

28. (3/8) Prove that $q_n\colon S^1 \to S^1$ defined by $q_n(z) = z^n$ (where $z \in \mathbb{C}$) has unique path lifting and unique homotopy lifting. Hint: do not repeat the proof of these results for $p\colon \mathbb{R} \to S^1$. Define $p_n\colon \mathbb{R} \to S^1$ by $p_n(r) = p(r/n)$ and use the facts that $p = q_n \circ p_n$ and that p has unique path lifting and unique homotopy lifting.

    Let $\alpha\colon I \to S^1$ and suppose that $s_0 \in S^1$ with $p(s_0) = \alpha(0)$. Define $p_n\colon \mathbb{R} \to S^1$ by $p_n(r) = p(r/n)$, so that $q_n \circ p_n(r) = (e^{2\pi i r/n})^n = e^{2\pi i r} = p(r)$, that is, $q_n \circ p_n = p$. Choose $r_0 \in \mathbb{R}$ with $p_n(r_0)$. By unique lifting for p, there exists $\widetilde{\alpha}\colon I \to \mathbb{R}$ so that $p \circ \widetilde{\alpha} = \alpha$. Then, we have $p_n \circ \widetilde{\alpha}\colon I \to S^1$ with $q_n \circ p_n \circ \widetilde{\alpha} = p \circ \widetilde{\alpha} = \alpha$, and $p_n \circ \widetilde{\alpha}(0) = p_n(r_0) = s_n$, proving existence of lifts for q_n.

    For uniqueness, suppose that $\widetilde{\alpha_1}, \widetilde{\alpha_2}\colon I \to S^1$ are two lifts of α taking 0 to s_0. Define $s_n\colon \mathbb{R} \to \mathbb{R}$ by $s_n(r) = nr$, so that $p_n \circ s_n = p$ and $p_n \circ s_n(r_0/n) = s_0$. By uniqueness of lifts for p, for each of $i = 1, 2$ there exists a unique $\widetilde{\widetilde{\alpha}}_i\colon I \to \mathbb{R}$ for which $p \circ \widetilde{\widetilde{\alpha}}_i = \widetilde{\alpha}_i$ and $\widetilde{\widetilde{\alpha}}_i(0) = r_0/n$. Now $q_n \circ p_n \circ s_n \circ \widetilde{\widetilde{\alpha}}_i = q_n \circ p \circ \widetilde{\widetilde{\alpha_1}} = q_n \widetilde{\alpha}_i = \alpha$. That is, $p \circ s_n \circ \widetilde{\widetilde{\alpha_1}} = p \circ s_n \circ \widetilde{\widetilde{\alpha_2}}$. Since $s_n \circ \widetilde{\widetilde{\alpha}}_i(0) = s_n(r_0/n) = r_0$, uniqueness of lifts for p implies that $s_n \circ \widetilde{\widetilde{\alpha_1}} = s_n \circ \widetilde{\widetilde{\alpha_1}}$. Therefore $p_n \circ s_n \circ \widetilde{\widetilde{\alpha_1}} = p_n \circ s_n \circ \widetilde{\widetilde{\alpha_2}}$. But this says that $p \circ \widetilde{\widetilde{\alpha_1}} = p \circ \widetilde{\widetilde{\alpha_2}}$, which is $\widetilde{\alpha_1} = \widetilde{\alpha_2}$.

The same proof works for unique lifting of homotopies.

29. (3/8) Give an example of a map $p\colon E \to B$ that has path lifting and homotopy lifting, but not uniquely. Hint: one example carries $\mathbb{R} \times I$ to S^1.

    Define $q\colon \mathbb{R} \times I \to \mathbb{R}$ by $q(r,s) = p(r)$. Let $\alpha\colon I \to \mathbb{R}$ be any path, and let $(r_0, s_0) \in \mathbb{R} \times I$ with $q(r_0, s_0) = \alpha(0)$, which says that $p(r_0) = \alpha(0)$. By the existence of lifts for p, there exists $\widetilde{\alpha}\colon I \to \mathbb{R}$ such that $p \circ \widetilde{\alpha} = \alpha$ and $\widetilde{\alpha}(0) = r_0$. Let $\beta\colon I \to I$ be any path with $\beta(0) = s_0$. Define $\widetilde{\widetilde{\alpha}}\colon I \to \mathbb{R} \times I$ by $\widetilde{\widetilde{\alpha}}(t) = (\widetilde{\alpha}(t), \beta(t))$. Then $q \circ \widetilde{\widetilde{\alpha}} = p \circ \widetilde{\alpha} = \alpha$ and $\widetilde{\widetilde{\alpha}}(0) = (\widetilde{\alpha}(0), \beta(0)) = (r_0, s_0)$, so $\widetilde{\widetilde{\alpha}}$ is a lift of α starting at (r_0, s_0). But each different choice of β gives a different lift. So starting from any point in $q^{-1}(\alpha(0))$, we always have infinitely many lifts of α. The same construction works for homotopies, replacing I by $I \times I$.

30. Let A be a subspace of X, and $i\colon A \to X$ the inclusion map. Recall that a *retraction* $r\colon X \to A$ is a map such that $r \circ i = id_A$. Define r to be a deformation retraction if there is a homotopy $F\colon id_X \simeq i \circ r$ with $F(a,t) = a$ for all t and all $a \in A$. (Note: this is sometimes called a *strong* deformation retraction.) If there exists a deformation retraction from X to A, we say that A is a *deformation retract* of X.

    1. Show that each $X \times \{t_0\}$ is a deformation retract of $X \times I$ (most of it is just showing that each t_0 is a deformation retract of I).

       A deformation retract of I to $\{t_0\} \subset I$ is defined by $R(t,s) = (1-s)t + st_0$. Now, define $F\colon X \times I \to X$ by $F((x,t),s) = (x, R(t,s))$.

    2. Show that the center circle of a Möbius band is a deformation retract of the Möbius band.

       Regard the Möbius band M as the square $I \times I$ with identifications $(0,y) \sim (1, 1-y)$. Note that the center circle is the subset $(I \times \{1/2\})/(0,1/2) \sim (1,1/2)$. Define a deformation retraction $F\colon I \times I \to I \times I$ by $F((x,y),t) = (x, (1-t)y + t/2)$. This is a deformation retraction of $I \times I$ to $I \times \{1/2\}$. To check that it produces a well-defined map on $M \times I$, we observe that $F((0,y),t) = (0, (1-t)y + t/2) \sim (1, 1 - (1-t)y - t/2) = (1, (1-t)(1-y) + t/2) = F((1, 1-y), t)$ for all t, y. So F preserves identified points and therefore it induces a deformation retraction $\overline{F}\colon M \times I \to M$ onto the center circle.

3. Show that if A is a deformation retract of X, then $i_{\#}\colon \pi_1(A, a_0) \to \pi_1(X, a_0)$ is an isomorphism for each basepoint $a_0 \in A$.

   We will show that $i_{\#}\colon \pi_1(A, a_0) \to \pi_1(X, a_0)$ is an isomorphism. We have $id_A = r \circ i$, so $id_{\pi_1(A,a_0)} = r_{\#} \circ i_{\#}$, showing that $i_{\#}$ is injective. To see that $i_{\#}$ is surjective, let $\langle \alpha \rangle \in \pi_1(X, a_0)$. Define $G\colon I \times I \to X$ by $G(t, s) = F(\alpha(t), s)$. Then $G(t, 0) = F(\alpha(t), 0) = \alpha(t)$, $G(0, s) = F(\alpha(0), s) = F(a_0, s) = a_0$ and similarly $G(1, s) = a_0$, and $G(t, 1) = F(\alpha(t), 1) = i \circ r(\alpha(t)) \in A$. Since $r \circ \alpha(t) \in A$, $i^{-1} \circ r \circ \alpha(t)$ is defined. Letting $\beta = i^{-1} \circ r \circ \alpha$, we have $i_{\#}(\langle \beta \rangle) = \langle i \circ \beta \rangle = \langle i \circ i^{-1} \circ r \circ \alpha \rangle = \langle r \circ \alpha \rangle = \langle \alpha \rangle$.