Math 5863 homework

33. (3/22) For $n \geq 2$, the *dihedral group of order* $2n$ is the group D_n consisting of all pairs $\alpha^i\beta^j$ where i is an integer modulo n and j is an integer modulo 2, with the multiplication rule that $\alpha^i\beta^j\alpha^k\beta^\ell = \alpha^{i+(-1)^j k}\beta^{j+\ell}$ (that is, $\beta\alpha^i\beta^{-1} = \alpha^{-i}$). Verify the following:

    1. Check that the condition $\alpha^i\beta^j\alpha^k\beta^\ell = \alpha^{i+(-1)^j k}\beta^{j+\ell}$ implies that $\beta\alpha\beta^{-1} = \alpha^{-1}$, and that the condition that $\beta\alpha\beta^{-1} = \alpha^{-1}$ implies that $\alpha^i\beta^j\alpha^k\beta^\ell = \alpha^{i+(-1)^j k}\beta^{j+\ell}$. Thus, people write $D_n = \langle \alpha, \beta \mid \alpha^n = \beta^2 = 1, \beta\alpha\beta^{-1} = \alpha^{-1}\rangle$.
    2. D_n has $2n$ elements.
    3. D_1 is isomorphic to C_2.
    4. D_2 is isomorphic to $C_2 \times C_2$.
    5. D_n is nonabelian for $n \geq 3$.
    6. The powers of α form a subgroup isomorphic to C_n.
    7. The powers of β form a subgroup isomorphic to C_2.
    8. Find the conjugacy class of each element of D_n.

34. (3/22) Recall that the group $\mathrm{Isom}_+(\mathbb{R}^2)$ of orientation-preserving isometries consists of all compositions $T_v R_\alpha$, for $v \in \mathbb{R}^2$ and $\alpha \in S^1$ (where we regard S^1 as the additive group of real numbers modulo 2π), with multiplication given by $T_v R_\alpha T_w R_\beta = T_{v+R_\alpha(w)} R_{\alpha+\beta}$. Note that the inverse of $T_v R_\alpha$ is $R_{-\alpha} T_{-v}$, which is also equal to $T_{R_{-\alpha}(-v)} R_{-\alpha}$.

    1. Verify that the conjugacy class of T_v ($v \neq 0$) is $\{T_w \mid \| w \| = \| v \|\}$. Describe these elements geometrically.
    2. Verify that the conjugacy class of R_α ($\alpha \neq 0$) is $\{T_v R_\alpha \mid v \in \mathbb{R}^2\}$. Show that these elements are exactly the isometries that rotate the plane through an angle α about some fixed point. (Observe that each conjugate can be written in the form $T_w R_\alpha T_{-w}$, and think about its geometric effect on the plane.)