Math 5863 homework solutions

23. Let G be a group. For an element g group G, define *conjugation by g* to be the function $\mu(g)\colon G \to G$ that sends x to gxg^{-1}.

   1. Check that $\mu(1) = id_G$ and $\mu(g_1 g_2) = \mu(g_1)\mu(g_2)$. Deduce that $\mu(g)$ is an isomorphism of G.

      $\mu(1)(x) = 1x1 = x = id_G(x)$ for all x. For all x, $\mu(g_1g_2)(x) = g_1g_2x(g_1g_2)^{-1} = g_1g_2xg_2^{-1}g_1^{-1} = g_1\mu(g_2)(x)g_1^{-1} = \mu(g_1)\mu(g_2)(x)$, so $\mu(g_1g_2) = \mu(g_1)\mu(g_2)$. Conjugation is a homomorphism because $\mu(g)(g_1g_2) = gg_1g_2g^{-1} = gg_1g^{-1}gg_2g^{-1} = \mu(g)(g_1)\mu(g)(g_2)$. Using the facts already shown, we have $id_G = \mu(1) = \mu(gg^{-1}) = \mu(g)\mu(g^{-1})$, and similarly $id_G = \mu(g^{-1})\mu(g)$, so $\mu(g)$ is an automorphism with inverse $\mu(g^{-1})$. That is, $\mu(g)^{-1} = \mu(g^{-1})$.

   2. Define $\mathrm{Aut}(G)$ to be the set of automorphisms of G. Check that $\mathrm{Aut}(G)$ is a group under the operation of composition.

   3. Define $\mathrm{Inn}(G)$ to be the set of inner automorphisms of G. Check that $\mathrm{Inn}(G)$ is a normal subgroup of G.

      An inner automorphism is a conjugation $\mu(g)$. The inner automorphisms form a subgroup $\mathrm{Inn}(G)$ of $\mathrm{Aut}(G)$ because if $\mu(g_1), \mu(g_2) \in \mathrm{Inn}(G)$, then $\mu(g_1)\mu(g_2)^{-1} = \mu(g_1)\mu(g_2^{-1}) = \mu(g_1g_2^{-1}) \in \mathrm{Inn}(G)$. To show $\mathrm{Inn}(G)$ is normal, let $\phi \in \mathrm{Aut}(G)$ and $g \in G$. We have $\phi\mu(g)\phi^{-1}(x) = \phi\mu(g)(\phi^{-1}(x)) = \phi(g\phi^{-1}(x)g^{-1}) = \phi(g)x\phi(g^{-1}) = \phi(g)x\phi(g)^{-1} = \mu(\phi(g))(x)$, so $\phi\mu(g)\phi^{-1} = \mu(\phi(g)) \in \mathrm{Inn}(G)$.

24. Let $p_1\colon X \times Y \to X$ and $p_2\colon X \times Y \to Y$ denote the projections. Show that $(p_1)_\# \times (p_2)_\#\colon \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ is an isomorphism.

25. Let x_0 and x_1 be two points in the same path-component of X. For a path $\gamma\colon I \to X$ from x_0 to x_1, define $h_\gamma\colon \pi_1(X, x_1) \to \pi_1(X, x_0)$ by $h_\gamma(\langle\alpha\rangle) = \langle\gamma * \alpha * \overline{\gamma}\rangle$.

   1. Show that h_γ is a well-defined homomorphism.

      Suppose that $\langle\alpha'\rangle = \langle\alpha\rangle$, so that $\alpha \simeq_p \alpha'$. Then we have $h_\gamma(\langle\alpha\rangle) = \langle\gamma * \alpha * \overline{\gamma}\rangle$ and $h_\gamma(\langle\alpha'\rangle) = \langle\gamma * \alpha' * \overline{\gamma}\rangle$. By properties of $\simeq_p$ developed in class, $\gamma * \alpha * \overline{\gamma} \simeq_p \gamma * \alpha' * \overline{\gamma}$, so $h_\gamma(\langle\alpha\rangle) = \langle$ and $h_\gamma(\langle\alpha'\rangle) = \langle$ are equal elements of $\pi_1(X, x_0)$.

   2. Show that if $\gamma(1) = \tau(0)$, then $h_{\gamma*\tau} = h_\gamma h_\tau$.

      For any $\langle\alpha\rangle \in \pi_1(X, \tau(1))$, we have $h_{\gamma*\tau}(\langle\alpha\rangle) = \langle\gamma * \tau * \alpha * \overline{\gamma * \tau}\rangle = \langle\gamma * \tau * \alpha * \overline{\tau} * \overline{\gamma}\rangle = h_\gamma(\langle\tau * \alpha * \overline{\tau}\rangle) = h_\gamma h_\tau(\langle\alpha\rangle)$.

   3. Show that if γ is a loop at x_1, then $h_\gamma = \mu(\langle\gamma\rangle)$.

      For any $\langle\alpha\rangle \in \pi_1(X, x_1)$, we have $h_\gamma(\langle\alpha\rangle) = \langle\gamma * \alpha * \overline{\gamma}\rangle = \langle\gamma\rangle\langle\alpha\rangle\langle\overline{\gamma}\rangle = \langle\gamma\rangle\langle\alpha\rangle\langle\gamma\rangle^{-1} = \mu(\langle\gamma\rangle)(\langle\alpha\rangle)$.

   4. Show that if $\gamma_1 \simeq_p \gamma_2$, then $h_{\gamma_1} = h_{\gamma_2}$. Deduce that h_γ is an isomorphism with inverse $h_{\overline{\gamma}}$. Thus, $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ are isomorphic as long as x_0 and x_1 are in the same path component of X.

If $\gamma_1 \simeq_p \gamma_2$, then $\overline{\gamma_1} \simeq \overline{\gamma_2}$, and for any $\langle \alpha \rangle \in \pi_1(X, x_1)$, we have $\gamma_1 * \alpha * \overline{\gamma_1} \simeq_p \gamma_2 * \alpha * \overline{\gamma_2}$, so $h_{\gamma_1}(\langle \alpha \rangle) = h_{\gamma_2}(\langle \alpha \rangle)$. Noting that h_{c_x} is the identity on $\pi_1(X, x)$, and using the previous part, we have for any loop α at x_1 that $\langle \alpha \rangle = h_{c_{x_1}}(\langle \alpha \rangle) = h_{\overline{\gamma} * \gamma}(\langle \alpha \rangle) = h_{\overline{\gamma}} h_\gamma(\langle \alpha \rangle)$. Since this implies also that $h_\gamma h_{\overline{\gamma}} = h_{\overline{\overline{\gamma}}} h_{\overline{\gamma}}$ is the identity on $\pi_1(X, x_0)$, we deduce that h_γ and $h_{\overline{\gamma}}$ are inverse homomorphisms, so are isomorphisms.

5. Show that if β is another path from x_0 to x_1, then $h_\beta^{-1} \circ h_\alpha = \mu(\langle \overline{\beta} * \alpha \rangle)$.

   Using the previous parts, we find that $h_\beta^{-1} h_\alpha = h_{\overline{\beta}} h_\alpha = h_{\overline{\beta} * \alpha} = \mu(\langle \overline{\beta} * \alpha \rangle)$.

6. Deduce that if $\pi_1(X, x_1)$ is abelian, then $h_\alpha \colon \pi_1(X, x_1) \to \pi_1(X, x_0)$ is independent of the choice of path α from x_0 to x_1.

   If $\pi_1(X, x_1)$ is abelian, then $h_\beta^{-1} h_\alpha = \mu(\langle \overline{\beta} * \alpha \rangle) = id_{\pi_1(X, x_1)}$, so $h_\beta = h_\alpha$.

As a consequence of the previous problem, all choices of α give the same isomorphism h_α when X is path-connected and $\pi_1(X, x_0)$ is abelian. That is, there is a way to identify $\pi_1(X, x_0)$ with $\pi_1(X, x_1)$ that is independent of all choices, so in this situation, one may safely write $\pi_1(X)$.