Math 5863 homework

31. (3/22) Denote the automorphism group of a group G by $\mathrm{Aut}(G)$. Determine the following automorphism groups:

    1. $\mathrm{Aut}(\mathbb{Z})$. (Consider $\phi(1)$.)
    2. C_n. (Write C_n as $\{1, \alpha, \alpha^2, \ldots, \alpha^{n-1}\}$. Observe that a homomorphism $\phi\colon C_n \to C_n$ is completely determined by $\phi(\alpha) = \alpha^m$. Show that ϕ is injective– hence bijective, since C_n is finite— if and only if m and n are relatively prime. Deduce that $\mathrm{Aut}(C_n) \cong \{1 \leq m < n \mid \gcd(m,n) = 1\}$ with the operation of multiplication modulo n.)
    3. Verify that $\mathrm{Aut}(C_{12}) \cong C_2 \times C_2$.
    4. $\mathrm{Aut}(\mathbb{Z} \times \mathbb{Z})$ (Regard elements of $\mathbb{Z} \times \mathbb{Z}$ as column vectors $\begin{bmatrix} a \\ b \end{bmatrix}$. Write $\phi\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \left(\begin{bmatrix} a \\ c \end{bmatrix}\right)$ and $\phi\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \left(\begin{bmatrix} b \\ d \end{bmatrix}\right)$, and observe that ϕ equals left multiplication by the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Verify that $\mathrm{Aut}(\mathbb{Z} \times \mathbb{Z})$ is isomorphic to the group $\mathrm{GL}(2, \mathbb{Z})$ of 2×2 matrices with integer entries and determinant ± 1. This generalizes to direct products of any number of copies of $\mathbb{Z}$, that is, $\mathrm{Aut}(\mathbb{Z}^n) \cong \mathrm{GL}(n, \mathbb{Z})$, but you do not need to work out the details of this.)

32. (3/22) Recall that two elements g_1 and g_2 of a group G are said to be *conjugate* if there exists an element $g \in G$ such that $g g_1 g^{-1} = g_2$. The *conjugacy class* of g_1 is the set of all elements of G that are conjugate to g_1.

    1. Verify that the relation of being conjugate is an equivalence relation.
    2. Verify that the conjugacy class of the identity element is the identity element.