Math 5863 homework

26. (3/8) Let $\alpha\colon I \to S^1$ be a path. Let $\widetilde{\beta}_1$ and $\widetilde{\beta}_2$ be two lifts of α to $\mathbb{R}$. Prove that for some $N \in \mathbb{Z}$, $\widetilde{\beta}_2(t) = \widetilde{\beta}_1(t) + N$ for all $t \in I$ (let $N = \widetilde{\beta}_2(0) - \widetilde{\beta}_1(0)$ and define $\tau(r) = r + N$, check that $p \circ \tau = p$, and use uniqueness of path lifting). Deduce that $\widetilde{\beta}_1(1) - \widetilde{\beta}_1(0) = \widetilde{\beta}_2(1) - \widetilde{\beta}_2(0)$.

27. (3/8) Prove that $q\colon \mathbb{Z} \times \mathbb{R} \to S^1$ defined by $q(n, r) = p(r)$ has unique path lifting and unique homotopy lifting. (Let $\alpha\colon I \to S^1$ and let $(n, r_0) \in \mathbb{Z} \times \mathbb{R}$ with $q(n, r_0) = \alpha(0)$. By unique path lifting for $\mathbb{R} \to S^1$, there exists $\widetilde{\alpha}_1\colon I \to \mathbb{R}$ with $p \circ \widetilde{\alpha}_1(t) = \alpha(t)$. Use $\widetilde{\alpha}_1$ to define the lift $\widetilde{\alpha}$. To prove that the $\widetilde{\alpha}$ is unique, let $p_1\colon \mathbb{Z} \times \mathbb{R} \to \mathbb{Z}$ and $p_2\colon \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ be the projection maps, and show that $p_1 \circ \widetilde{\alpha}$ and $p_2 \circ \widetilde{\alpha}$ are uniquely determined.)

28. (3/8) Prove that $q_n\colon S^1 \to S^1$ defined by $q_n(z) = z^n$ (where $z \in \mathbb{C}$) has unique path lifting and unique homotopy lifting. Hint: do not repeat the proof of these results for $p\colon \mathbb{R} \to S^1$. Define $p_n\colon \mathbb{R} \to S^1$ by $p_n(r) = p(r/n)$ and use the facts that $p = q_n \circ p_n$ and that p has unique path lifting and unique homotopy lifting.

29. (3/8) Give an example of a map $p\colon E \to B$ that has path lifting and homotopy lifting, but not uniquely. Hint: one example carries $\mathbb{R} \times I$ to S^1.

30. (3/8) Let A be a subspace of X, and $i\colon A \to X$ the inclusion map. Recall that a *retraction* $r\colon X \to A$ is a map such that $r \circ i = id_A$. Define r to be a deformation retraction if there is a homotopy $F\colon id_X \simeq i \circ r$ with $F(a, t) = a$ for all t and all $a \in A$. (Note: this is sometimes called a *strong* deformation retraction.) If there exists a deformation retraction from X to A, we say that A is a *deformation retract* of X.

    1. Show that each $X \times \{t_0\}$ is a deformation retract of $X \times I$ (most of it is just showing that each t_0 is a deformation retract of I).
    2. Show that the center circle of a Möbius band is a deformation retract of the Möbius band.
    3. Show that if A is a deformation retract of X, then $i_\#\colon \pi_1(A, a_0) \to \pi_1(X, a_0)$ is an isomorphism for each basepoint $a_0 \in A$.