Math 5863 homework solutions

17. Use the Classification Theorem to deduce the following facts about the Euler characteristic of a (compact, connected) 2-manifold F.

    1. $\chi(F) \leq 2$.
    2. $\chi(F) = 2$ if and only if $F = S^2$.
    3. $\chi(F) = 1$ if and only if F is a disk or a projective plane.
    4. $\chi(F) = 0$ if and only if F is an annulus, Möbius band, torus, or Klein bottle.
    5. Find all F with $\chi(F) = -1$.
    6. Find all F with $\chi(F) = -2$.

    $2 = \chi(F) = 2 - n - 2g - \ell$ implies that $n = g = \ell = 0$, so $F = S^2$.

    $1 = \chi(F) = 2 - n - 2g - \ell$ implies that $g = 0$, and either $n = 1$ and $\ell = 0$, giving $F = P$, or $n = 0$ and $\ell = 1$, giving $F = D$.

    Assume that $0 = \chi(F)$. Suppose first that F is orientable. Then we have $2 - 2g - \ell = 0$, so (g, ℓ) must be either $(1, 0)$ or $(0, 2)$, giving respectively $F = T$ or $F = D\#D$. If F is nonorientable, then we have $2 - n - \ell = 0$, so (n, ℓ) is one of $(2, 0)$, $(1, 1)$, or $(0, 2)$, giving respectively $F = P\#P$ (the Klein bottle), $F = P\#D$ (the Möbius band), or $(0, 2)$, giving $F = D\#D = S^1 \times I$ (the annulus).

    Assume that $-1 = \chi(F)$. Again, suppose that F is orientable. From $-1 = 2 - 2g - \ell$, we have $g \leq 1$. If $g = 1$, then $\ell = 1$, giving $F = T\#D$. If $g = 0$, then $\ell = 3$, giving $F = D\#D\#D$. Now suppose that F is nonorientable. From $-1 = 2-n-\ell$, we see that (n, ℓ) is one of $(3, 0)$, $(2, 1)$, or $(1, 2)$, giving respectively F to be $P\#P\#P$, $P\#P\#D$, or $P\#D\#D$, for a total of five surfaces.

    Assume that $-2 = \chi(F)$. Suppose first that F is orientable. From $-2 = 2-2g-\ell$, we must have $g \leq 2$, and the three possibilities $(g, \ell) = (2, 0)$, $(1, 2)$, or $(0, 4)$ give F respectively to be $T\#T$, $T\#D\#D$, or $D\#D\#D\#D$. Suppose now that F is nonorientable. From $-2 = 2-n-\ell$, we have the four possibilites for (n, ℓ) of $(4, 0)$, $(3, 1)$, $(2, 2)$, or $(1, 3)$, giving respectively $F = P\#P\#P\#P$, $F = P\#P\#P\#D$, $F = P\#P\#D\#D$, or $F = P\#D\#D\#D$, for a total of seven surfaces.

18. For each of the surfaces shown on the next page, use orientability and Euler characteristic to determine the homeomorphism type of the surface. The answer may depend on whether m is even or odd.

    For the first surface, the Euler characteristic is $2-m$ and the surface is orientable. When m is odd, there is one boundary circle, so we have $2 - m = 2 - 2g - 1$, giving $g = (m - 1)/2$ and $F = (m - 1)/2\ T\#D$. When m is even, there are two boundary circles and we have $2 - m = 2 - 2g - 2$, giving $g = (m - 2)/2$ and $F = (m - 2)/2\ T\#D\#D$.

The second surface has Euler characteristic $m - 2m = -m$, and has m boundary circles. We observe that when m is even, F is orientable and has m boundary circles, and an Euler characteristic gives $F = T\#mD$. When m is odd, F is nonorientable, and we find that $F = P\#P\#mD$.

The third surface has Euler characteristic $1 - m$ and is orientable. When m is odd, there are two boundary circles and we find that $F = (m-1)/2\; T\#D\#D$. When m is even, there is one boundary circle and $F = (m+2)/2\; T\#D$.