Math 5863 homework solutions

8. (2/1) Suppose that $h_0, h_1 \colon X \to Y$ are isotopic. Prove that if $g \colon Y \to Z$ is a homeomorphism, then $g \circ h_0 \simeq g \circ h_1$. Prove that if $k \colon Z \to X$ is a homeomorphism, then $h_0 \circ k \simeq h_1 \circ k$.

   Let h_t be the isotopy. Each $g \circ h_t$ is a homeomorphism, so $g \circ h_t$ defines an isotopy from $g \circ h_0$ to $g \circ h_1$, and similarly $h_t \circ k$ defines an isotopy from $h_0 \circ k$ to $h_1 \circ k$. If one wishes to worry about the continuity, just note that the isotopy $g \circ h_t$ is the composition of the map $X \times I \to Y \times I$, defined by sending (x, t) to $(h_t(x), t)$ (which is continuous since h_t is an isotopy) and the map $g \circ \pi_1 \colon Y \times I \to Z$.

9. (2/1) An imbedding $j \colon I \to I$ is called *order-preserving* if $j(0) < j(1)$, otherwise it is called *order-reversing.*

   1. Prove that if j is order-preserving, then $j(x_1) < j(x_2)$ whenever $x_1 < x_2$.

      Suppose that $x_1 < x_2$ but $j(x_1) > j(x_2)$. If $j(0) < j(x_2)$, then the Intermediate Value Theorem applied to the interval $[0, x_1]$ produces $c < x_1$ with $j(c) = j(x_2)$. If $j(x_2) < j(0) < j(x_1)$, then the Intermediate Value Theorem applied to the interval $[x_1, x_2]$ produces $c > x_1$ with $j(0) = j(c)$. If $j(x_1) < j(0)$, then the Intermediate Value Theorem applied to the interval $[x_2, 1]$ produces $c > x_2$ with $j(x_1) = j(c)$. In any case, we contradict the fact that j is an imbedding.

   2. Prove that there are exactly two isotopy classes of imbeddings of I into I, by showing that $j_0, j_1 \colon I \to I$ are isotopic if and only if they are both order-preserving or both order-reversing.

      Suppose first that j_0 and j_1 are both order-preserving. Define $j \colon I \times I \to I$ by $j(x, t) = (1 - t)j_0(t) + tj_1(t)$. Since j is clearly continuous, and $j(x, 0) = j(0)$ and $j(x, 1) = j(1)$, it suffices to show that each j_t is injective. Then, since I is compact Hausdorff, we will know automatically that each j_t is an imbedding, so j will be an isotopy of imbeddings from j_0 to j_1.

      By the previous part of the problem, we know that if $x_1 < x_2$, then $j_0(x_1) < j_0(x_2)$ and $j_1(x_1) < j_1(x_2)$. Suppose that for some t and some $x_1 < x_2$ we have $j_t(x_1) = j_t(x_2)$. Using the definition of j_t shows that $(1-t)(j_0(x_2)-j_0(x_1)) = -t(j_1(x_2)-j_1(x_1))$. This is a contradiction, since the left-hand side is positive and the right-hand side is negative.

      Suppose now that j_0 and j_1 are both order-reversing. Let $\rho \colon I \to I$ be the reflection homeomorphism defined by $\rho(x) = 1 - x$. Then $j_0 \circ \rho$ and $j_1 \circ \rho$ are order-preserving, so the previous case shows that $j_0 \circ \rho$ and $j_1 \circ \rho$ are isotopic. Problem 8 shows that $j_0 \circ \rho \circ \rho$ and $j_1 \circ \rho \circ \rho$ are isotopic. Since $\rho \circ \rho = id_I$, this shows that j_0 and j_1 are isotopic.

      Finally, we must show that if j_0 is order-preserving and j_1 is order-reversing, then j_0 is not isotopic to j_1. If j_t were an isotopy, then the Intermediate Value Theorem applied to the function $j_t(0)-j_t(1)$ produces a t_0 with $j_{t_0}(0) = j_{t_0}(1)$, contradicting the fact that j_{t_0} is an imbedding.

10. (2/1) Prove the Disk Lemma for $n = 1$ and $M = I$. That is, prove that if $j_1, j_2 \colon I \to I$ are imbeddings with image in the interior of I, then j_1 is ambiently isotopic to either j_2 or $j_2 \circ \rho$. Hint: this follows quickly from the fact that any two homeomorphisms of I are isotopic. Compose j_1 and/or j_2 by ρ to assume that both are order preserving. Extend the homeomorphism $j_2 \circ j_1^{-1} \colon j_1(I) \to j_2(I)$ to an order-preserving homeomorphism $h \colon I \to I$, by using linear maps on $\overline{I - j_1(I)}$. Now make use of the fact that id_I and h are isotopic.

    Assume first that both j_1 and j_2 are order-preserving. We have a homeomorphism $j_2 \circ j_1^{-1} \colon j_1(I) \to j_2(I)$. We extend this to a homeomorphism $h \colon I \to I$ using order-preseving linear homeomorphisms from $[0, j_1(0)]$ to $[0, j_2(0)]$ and from $[j_1(1), 1]$ to $[j_2(1), 1]$. Explicitly, for $x \in [0, j_1(0)]$ put $h(x) = xj_1(0)/j_2(0)$ (using the fact that both $j_1(0), j_2(0) > 0$), and for $x \in [j_1(1), 1]$ put $h(x) = 1-j_2(1)+(x-j_1(1))/(1-j_1(1))$ (using the fact that $j_1(1), j_2(1) < 1$). Since h is an order-preserving homeomorphism, there is an isotopy H from id_I to h. Then, $H_0 = id_I$ and $H_1 \circ j_1 = h \circ j_1 = j_2 \circ j_1^{-1} \circ j_1 = j_2$, showing that j_1 and j_2 are ambiently isotopic.

11. (2/1) A compact (connected) surface F is called *planar* if $F \neq S^2$ and F can be imbedded into S^2. Show that if F_1 and F_2 are planar, then the connected sum $F_1 \# F_2$ is planar. Hint: Let $D_1 \subset F_1$ and $D_2 \subset F_2$ be admissible disks. Use the Disk Lemma to show that there is an imbedding of F_1 in S^2 that carries D_1 to the upper hemisphere, and there is an imbedding of F_2 in S^2 that carries D_2 to the lower hemisphere.

    Fix imbeddings $f_1 \colon F_1 \to S^2$ and $f_2 \colon F_2 \to S^2$. Let $i_+ \colon D^2 \to S^2$ be the imbedding $i_+(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$, and let $i_- \colon D^2 \to S^2$ be the imbedding $i_-(x, y) = (x, y, -\sqrt{1 - x^2 - y^2})$. Finally, let $i_1 \colon D^2 \to F_1$ and $i_2 \colon D^2 \to F_2$ be any (admissible) imbeddings. By the Disk Lemma, there are ambient isotopies J_1 and J_2 of S^2 with $(J_1)_1 \circ f_1 \circ i_1 = i_+ \circ \rho^{\epsilon_1}$ and $(J_2)_1 \circ f_2 \circ i_2 = i_- \circ \rho^{\epsilon_2}$, where each of ϵ_1 and ϵ_2 is either 0 or 1. Denote $(J_i)_1 \circ f_i$ by k_i. Notice that $i_+(D^2) = i_+ \circ \rho^{\epsilon_1}(D^2) = k_1(i_1(D^2))$, so $i_+(D^2)$ is in the interior of $k_1(F_1)$, and similarly $i_-(D^2)$ is in the interior of $k_2(F_2)$.

    We may define $F_1 \# F_2$ using any (admissible) imbeddings of D^2 into F_1 and F_2, so assume it is defined using the imbeddings $j_1 = k_1^{-1} \circ i_+ \colon D^2 \to F_1$ and $j_2 = k_2^{-1} \circ i_- \colon D^2 \to F_2$. That is, let $F_1 \# F_2$ be the identification space obtained from the union of $F_1 - j_1(\mathrm{int}(D^2))$ and $F_1 - j_2(\mathrm{int}(D^2))$ by identifying $j_1(p)$ with $j_2(p)$ for each $p \in \partial D^2$. Now, $F_1 \# F_2 \neq S^2$, for if so then when we cut S^2 along ∂D^2, we obtain two D^2's, and then each of F_1 and F_2 would have to be S^2. It remains to show that $F_1 \# F_2$ can be imbedded in S^2. Define $f \colon F_1 \# F_2 \to S^2$ by $f(z) = k_1(z)$ for $z \in F_1 - j_1(\mathrm{int}(D^2))$ and $f(z) = k_2(z)$ for $z \in F_2 - j_2(\mathrm{int}(D^2))$. For $p \in \partial D^2$, we have $f(j_1(p)) = k_1 \circ j_1(p) = k_1 \circ k_1^{-1} \circ i_+(p) = i_+(p)$ and $f(j_2(p)) = k_2 \circ j_2(p) = k_2 \circ k_2^{-1} \circ i_-(x) = i_-(p)$. Since $i_+(p) = i_-(p)$ for $p \in \partial D^2$, this shows f is well-defined. It is continuous by gluing on closed sets, and is injective by construction. Since it is a continuous injection from a compact space into a Hausdorff space, it is an imbedding. Therefore $F_1 \# F_2$ is planar.