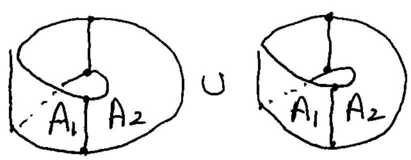
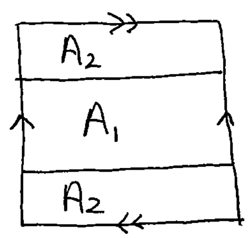
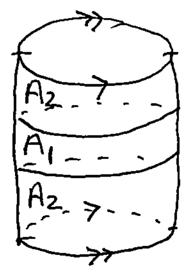
Math 5863 homework solutions

Instructions: All problems should be prepared for presentation at the problem sessions. If a problem has a due date listed, then it should be written up formally and turned in on the due date.

1. (1/18) The Klein bottle K can be constructed from two annuli A_1 and A_2 by identifying their boundaries in a certain way. For each of the three descriptions of K discussed in class (two Möbius bands with boundaries identified, the square with certain identifications on its boundary, and $S^1 \times I$ with the two ends identified), make a drawing showing where A_1 and A_2 appear in K.



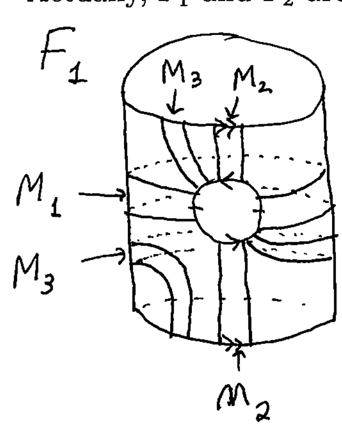


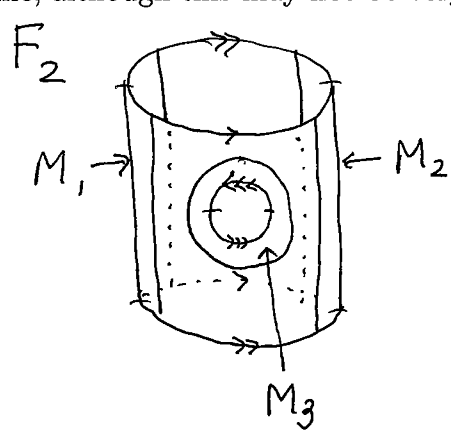


2. (1/18) Two surfaces F_1 and F_2 can be constructed as follows. Start with $S^1 \times I$, and remove the interior of a small disk D from the interior of $S^1 \times I$. For F_1, identify each $(\theta, 0)$ with $(\theta, 1)$ and identify each point of ∂D with its antipodal point (that is, if ∂D is regarded as S^1, then v is identified with $-v$). For F_2, identify each $(\theta, 0)$ with $(\overline{\theta}, 1)$ and identify each point of ∂D with its antipodal point.

   1. Make drawings illustrating each of F_1 and F_2. Notice that both are closed surfaces.
   2. Find three disjoint Möbius bands imbedded in F_1.
   3. Find three disjoint Möbius bands imbedded in F_2.

   Actually, F_1 and F_2 are homeomorphic, although this may not be very easy to see.





3. (1/18) Let M and N be n-dimensional manifolds, and let U be an open subset of M. Suppose that $f\colon U \to N$ is a continuous injection. Prove that f takes open sets in U to open sets in N.

Solution 1: Let V be an open subset of U. It suffices to show that for any point $f(x)$, there exists an open set J in N with $f(x) \in J \subseteq f(V)$. Choose an open neighborhood $W_{f(x)}$ of $f(x)$ with $W_{f(x)}$ homeomorphic to $\mathbb{R}^n$. Choose an open neighborhood V_x of x with $V_x \approx \mathbb{R}^n$. Now $f^{-1}(W_{f(x)})$ is an open neighborhood of x, and f carries the open subset $f^{-1}(W_{f(x)}) \cap V \cap V_x$ of x by a continuous injection into $W_{f(x)}$. By Invariance of Domain, $J = f(f^{-1}(W_{f(x)}) \cap U \cap V_x)$ is open in $W_{f(x)}$, and hence in N, and $f(x) \in J \subseteq W_{f(x)}$.

Solution 2: Let V be open in U. It is an open subset of a manifold, hence is a manifold, so we may select a collection $\{V_\alpha\}$ of charts whose union is V. Let $\{W_\beta\}$ be a collection of charts whose union is N. We have

$$\begin{aligned} f(V) &= \cup_\alpha f(V_\alpha) = (\cup_\alpha f(V_\alpha)) \cap (\cup_\beta W_\beta) \\ &= \cup_\alpha (f(V_\alpha) \cap (\cup_\beta W_\beta)) = \cup_{\alpha,\beta} f(V_\alpha \cap f^{-1}(W_\beta)) \end{aligned}$$

Each $V_\alpha \cap f^{-1}(W_\beta)$ is open in V_α, so by Invariance of Domain its image $f(V_\alpha \cap f^{-1}(W_\beta))$ is an open subset of W_β, and hence is open in N. So $f(V)$ is a union of open subsets of N.