Math 5863 homework

39. (4/5) Let X, Y, and Z be spaces, with Y locally compact Hausdorff, and give all sets of continuous functions the C-O topology. Define the composition function $C(f,g)\colon \mathcal{C}(X,Y)\times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z)$ by $C(f,g) = g\circ f$. Prove that C is continuous. (Take as known the fact that if K is a compact subset of a locally compact Hausdorff space, and W is an open set containing K, then there exists an open set V with $\overline{V}$ compact and $K \subseteq V \subseteq \overline{V} \subseteq W$.)

40. (4/14) Suppose that $f\colon (X,x_0) \to (Y,y_0)$ and $g\colon (X,x_0) \to (Y,y_1)$ are maps, and $H\colon f \simeq g$. Define a path $\gamma\colon I \to Y$ by $\gamma(t) = H(x_0,t)$. Show informally that $f_\# = h_\gamma \circ g_\#$. Deduce that if $y_0 = y_1$, then $f_\#$ and $g_\#$ differ by an inner automorphism of $\pi_1(Y,y_0)$.

41. (4/14) Prove that a space X is contractible if and only if X is homotopy equivalent to a space with one point.

42. (4/14) Let $X = S^1 \times D^2$, let $p_0 = (1,0) \in S^1$. Let C_1 be the circle consisting of all (θ, p_0), and let C_2 be the circle consisting of all (p_0,θ). Prove that the inclusion $C_1 \to X$ is a homotopy equivalence, but the inclusion $C_2 \to X$ is not a homotopy equivalence.

43. (4/14) Let $p\colon E \to B$ be a covering map, with B path-connected.

    1. Prove that if $b_0, b_1 \in B$, then $p^{-1}(b_0)$ and $p^{-1}(b_1)$ have the same cardinality. Hint: For a path α in B, define $\Phi_\alpha\colon p^{-1}(\alpha(0)) \to p^{-1}(\alpha(1))$ as follows. For $e \in p^{-1}(\alpha(0))$, let $\widetilde{\alpha}$ be the lift of α starting at e, and put $\Phi(e) = \widetilde{\alpha}(1)$. Prove that $\Phi_{\overline{\alpha}} \circ \Phi_\alpha$ is the identity on $p^{-1}(\alpha(0))$. Deduce that Φ_α is a bijection.
    2. Prove that if E is compact, then p is finite-to-one.