Mathematics 4513-001 Name (please print) ______________________

Final Examination

December 19, 2008

Instructions: Give brief, clear answers. If asked for a definition, give the definition that we have used in this course. In some of the problems, you will need to calculate using the formula $\Omega_\ell X = X - 2\langle X, P\rangle P$ for reflection of S^2 across the line with pole P.

I. (6) (i) Define a *line* in $\mathbb{R}^2$.

(ii) Using the notation of the definition, write an expression for the line thorough $(1, 0)$ and $(3, 5)$.

(iii) Find a unit normal to this line in (ii).

II. (5) Let X be a set with a distance function $d\colon X \times X \to X$. Let f and g be functions from X to X.

(i) Define what it means to say that f is an *isometry* of X.

(ii) Prove that if f and g are isometries, then their composition fg is also an isometry.

III. (7) Recall that for $a, b \in \mathbb{R}^3$ we defined the cross product $a \times b$ to be the unique vector in $\mathbb{R}^3$ such that $\langle a \times b, x\rangle = \det(x, a, b)$, where $\det(x, a, b)$ is the determinant of the matrix whose rows are x, a, and b. Use this definition to verify the following facts about the cross product. You may use standard properties of the determinant, and may use the fact that if $\langle a, x\rangle = \langle b, x\rangle$ for all x, then $a = b$.

(i) $a \times b = -b \times a$

(ii) $\langle a \times b, c\rangle = \langle a, b \times c\rangle$

(iii) If $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$, then $a \times b = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$.

IV. (6) Recall that a line in S^2 with pole P is defined to be the set $\{X \in S^2 \mid \langle X, P\rangle = 0\}$. Let P and Q be distinct points in S^2 with $P \neq -Q$.

(i) Give an expression (in terms of P and Q) for a pole of the line ℓ that contains P and Q.

(ii) Give an expression (in terms of P and Q) for a pole of the line perpendicular to ℓ that contains Q.

V. (9) Let G be the group whose elements are points in the plane $\mathbb{R}^2$, and whose operation is the usual vector addition, $(x_1, y_1) + (x_2, y_2) = (x_1 + y_1, x_2 + y_2)$. Define $\Phi\colon G \to G$ by $\Phi(X) = (3x, -2y)$, where $X = (x, y)$. Verify that Φ is an isomorphism (that is, verify that Φ is injective, surjective, and satisfies $\Phi(X_1 + X_2) = \Phi(X_1) + \Phi(X_2)$).

VI. (5) (i) Calculate the determinants of the matrices $\mathrm{rot}(\theta)$ and $\mathrm{ref}(\theta)$.

(ii) Explain geometrically why these are the values one would expect for these determinants.

VII. (7) (i) Give the definitions (in terms of reflections), of a translation and of a rotation of the *plane.*

(ii) Explain why these concepts become the *same* concept when we are working on S^2.

VIII. (6) (i) There is a version of the Three Reflections Theorem for the geometry of S^2. Tell what it says.

(ii) Use the Three Reflections Theorem to show that if α, β, and ℓ are lines in S^2 that meet at a point P, then there is a line m through P so that $\Omega_\alpha\Omega_\beta = \Omega_\ell\Omega_m$.

IX. (5) Recall that we define the distance function in S^2 by $d(X,Y) = \cos^{-1}(\langle X,Y\rangle)$. Verify that J is an isometry of S^2 if and only if J preserves the inner product (that is, if and only if $\langle JX,JY\rangle = \langle X,Y\rangle$ for all X and Y in S^2).

X. (6) Let α, β, and γ be lines in S^2 with α and γ both perpendicular to β.

(i) Sketch a picture of these three lines in S^2, oriented so that the intersection points of α and γ appear as the north and south poles.

(ii) Suppose that $\{e_1, e_2, e_3\}$ is an orthonormal basis for which e_3 is an intersection point of α and γ, say the north pole in part (i). Write the general form of the matrices of Ω_α, Ω_β, and Ω_γ with respect to this basis (you do not need to do any calculations, just use your knowledge of how the matrices of these reflections look for this kind of basis). It is a good idea to write them in block form, so that even though they are 3×3 matrices, they appear visually as 2×2 matrices.

(iii) By considering the product of the three matrices in part (ii), show that the composition $\Omega_\alpha\Omega_\beta\Omega_\gamma$ is a glide reflection.

XI. (4) Let $\{E_1, E_2, E_3\}$ be the standard basis of $\mathbb{R}^3$, and let J be an isometry of S^2. Use the Orthonormal Basis Theorem to express $J(X)$ in terms of E_1, E_2, and E_3.

XII. (5) Let P, Q, and R be points in S^2, with $P \neq -Q$. Explain why P, Q, and R are collinear exactly when $\langle R, P\times Q\rangle = 0$.

XIII. (6) Let G be the group of isometries of S^2, and let H be the subgroup of G consisting of the isometries that take the point $(0,0,1)$ to itself, that is, the isometries J of S^2 with $J((0,0,1)) = (0,0,1)$.

(i) Tell what one would need to do to prove that H is not normal in G.

(ii) Prove that H is not normal in G.