Mathematics 2433-001 | Name (please print) ____________________

Final Examination Form A

December 13, 1999 | Student Number ____________________

I. (6) Write an equation for the plane that contains the point $(3, -3, -1)$ and the line with parametric equations $x = 2 + t$, $y = -t$, and $z = 13$. (Find a point on the line and take the vector from it to $(3, -3, -1)$ to get a vector in the plane. The direction vector of the line also lies in the plane. Use the two vectors to get a normal vector.)

II. (6) Write the Maclaurin series for e^t. Use it to write the Maclaurin series for e^{-t^2}. Use integration to find a series equal to $\sqrt{\frac{2}{\pi}} \int_0^x e^{-t^2}\, dt$. This is called the *error function,* it is important in the fields of statistics and partial differential equations.

III. (10) Consider the helix given by the parametric equations $x = a\cos(t)$, $y = a\sin(t)$, $z = bt$, or equivalently by the vector-valued function $\vec{r}(t) = a\cos(t)\vec{\imath} + a\sin(t)\vec{\jmath} + bt\vec{k}$.

1. Calculate the velocity $\vec{v}(t)$ and the acceleration $\vec{a}(t)$.

2. Verify that $\vec{v}(t)$ and $\vec{a}(t)$ are orthogonal.

3. Calculate the unit tangent vector $\vec{T}(t)$ and the unit normal vector $\vec{N}(t)$.

4. Use the formula $\kappa = \dfrac{\| \vec{r}''(t) \times \vec{r}'(t) \|}{\| \vec{r}'(t) \|^3}$ to calculate the curvature.

IV. (5) In this problem, all vectors are assumed to start at the origin. Draw two xyz-coordinate systems. On the first, draw a typical vector $\vec{v}$ of length between 2 and 3 that lies in the plane $x = 0$, has positive y-component, and negative z-component. Draw and label the cross products $\vec{\imath}\times\vec{v}$ and $\vec{v}\times\vec{\jmath}$. On the second coordinate system, draw $\vec{v}$ and $-\vec{\jmath}$, and draw and label the vector projection of $\vec{v}$ onto $-\vec{\jmath}$ and the vector projection of $-\vec{\jmath}$ onto $\vec{v}$.

V. (4) For the point whose xy-coordinates are $(-0.5, 0.45)$, find *all* possible polar coordinates (an inverse trigonometric function should appear in the answer).

VI. (10) Consider the curve C determined by the vector-valued function $\vec{r}(t) = \vec{\imath} + t^2\vec{\jmath} + t^3\vec{k}$. The point on C corresponding to $t = -1$ is $(1, 1, -1)$.

1. Calculate $\vec{r}\,'(t)$.

2. Write an equation *as a vector-valued function* $\vec{\ell}(t)$ for the tangent line to C at the point $(1, 1, -1)$.

3. Calculate the length of the portion of C that runs from $(1, 1, -1)$ to $(1, 0, 0)$.

4. Write *parametric equations* for a point that moves along the line which is tangent to C at the point $(1, 1, -1)$, so that at time $t = 0$ the point is at $(1, 1, -1)$, and so that the point moves *with constant speed equal to* 2.

VII. (10) For each of the following series, use the limit of the terms, or the comparison test, or the alternating series convergence test, or the ratio test to determine whether the series converges or diverges. Assume as known the convergence behavior of the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

1. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3 - 4}}$

2. $\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^{3/4}}$

3. $\sum_{n=1}^{\infty} \frac{n5^n}{3^{2n}}$

4. $\sum_{n=1}^{\infty} \frac{\tan(\frac{1}{n})}{n}$ (Hint: consider the graph of $y = \tan(x)$ and use it to compare $\tan(x)$ to a linear function for small positive x-values.)

VIII. (7) In spherical coordinates, graph the equation $\rho = 3$. On this surface, draw some of the lines where ϕ is constant, and some of the lines where θ is constant, indicating which is which. On another copy of the graph of $\rho = 3$, plot the point $(\rho, \theta, \phi) = (3, 0, \pi/4)$. Draw a path starting at this pint, that moves a short distance with increasing ϕ-coordinate, then moves a somewhat longer distance with increasing θ-coordinate, then moves with increasing ϕ-coordinate until $\phi = \pi/2$. On a third graph, sketch the region with $\rho \leq 3$ and $3\pi/4 \leq \phi \leq \pi$.

IX. (4) For a certain power series of the form $\sum_{n=0}^{\infty} c_n(x+1)^n$, it is known that the series converges when $x = -4$ and diverges when $x = 6$. Give upper and lower estimates for its radius of convergence R.

X. (6) Write the general formula for the Taylor series of a function $f(x)$ at $x = a$. Use it to calculate the Taylor series of the function $f(x) = x^4$ at $x = 1$.

XI. Consider the curve in the xy-plane given by the polar equation $r = \tan(\theta)$.
(8)

1. Use the formula $\dfrac{dy}{dx} = \dfrac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)}$ to find all values of θ for which the tangent line to the curve is horizontal.

2. Find expressions, as functions of θ, for x and y at the points on the curve. Find the limit of the x-coordinate as θ increases to $\pi/2$ and as θ decreases to $\pi/2$. Determine the behavior of the y-coordinate as θ increases to $\pi/2$ and as θ decreases to $\pi/2$.

3. Graph the curve for $0 \leq \theta \leq 3\pi/4$, $\theta \neq \pi/2$.